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# COMPLEXITY OF ELIAS ALGORITHM BASED ON CODES WITH COVERING RADIUS THREE

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The algorithm for finding the set of "nearest neighbors" in a set using compact blocks and hash functions is known (Elias algorithm). In this paper hash coding schemas associated to coverings by spheres of the same radius are considered. In general, such coverings can be obtained via perfect codes, and some other generalizations of perfect codes such as uniformly packed or quasi perfect codes. We consider the mentioned algorithm for Golay code and for two-error-correcting primitive BCH codes of lenght $2^m - 1$ for odd $m$. A formula of time complexity of the algorithm is obtained in these cases.

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**1. Introduction.** Let $E = \{0,1\}$. Consider the Cartesian power $E^n$, which is known as the set of vertices of $n$-dimensional unit cube. For $x, y \in E^n$ denote by $d(x,y)$ the Hamming distance between the vectors $x$ and $y$. For an $x \in E^n$ denote by $S_r^n(x)$ the sphere of radius $r$ centered at point $x$, i.e. $S_r^n(x) = \{y \in E^n / d(x,y) \leq r\}$ and denote by $O_r^n(x)$ the shell of radius $r$, i.e. $O_r^n(x) = \{y \in E^n / d(x,y) = r\}$. We will denote by $car(x)$ the carrier of the vector $x = (x_1, ..., x_n)$, i.e. $car(x) = \{i / x_i \neq 0, i \in \{1, ..., n\}\}$. Denote by $w(x)$ the weight of the vector $x$, i.e. $w(x) = \sum_{i=1}^n x_i$. We will call a code a nonempty subset $C$ of $E^n$ (usually some other additional properties take place such as linearity, cyclicity, etc.). The code $C$ is called linear, if $C$ is a linear subspace of $E^n$. Due to the binary nature of the spaces considered, $C$ is linear when $\forall c_1, c_2 \in C \Rightarrow c_1 + c_2 \in C$. Denote by $d_C$ the minimum distance of code $C$, i.e.

$$d_C = \min_{c_1, c_2 \in C, c_1 \neq c_2} d(c_1, c_2).$$

The packing radius [1] of code $C$ is called the following number: $d_C = \lfloor \frac{d_C - 1}{2} \rfloor$. Denote by $R_C$ the covering radius of the code $C$, i.e. $R_C = \max_{x \in E^n} \min_{c \in C} d(x,c)$. In the sequel, when it will make no confusion, we will use notations $d$,$r$ and $R$ instead of $d_C$, $r_C$ and $R_C$ respectively. We say that we have an $(n, M, d)R$ code $C$, if it has lenght $n$, cardinality $M$, distance $d$ and covering radius $R$. When it is known that $C$ is linear to fix that we use the notation $[n, k, d]R$,

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where $k$ is the dimension of $C$ as linear subspace. Recall that the code $C$ is called perfect [1], if $r_C = R_C$. It is known [1,2], that in binary spaces nontrivial perfect codes can have only the following two parameter sets.

I. $(2^m - 1, 2^{2^m - m - 1}, 3)1$,

II. $(23, 2^{11}, 7)3$,

where I corresponds to parameters of Hamming code, and II corresponds to parameters of Golay code. For $x \in E^n$ the coset of linear code $C$ is called the set $x + C = \{x + c/c \in C\}$. As it is known [1], two different cosets do not intersect and their union cover the space $E^n$. We denote by $G_C$ the generator matrix of the linear code $C$. Recall that $G_C$ is a matrix with rows forming basis of $C$. Let us denote by $H_C$ the parity check matrix of linear code $C$. If $C$ is $[n,k,d]R$ code, then $H_C$ is $(n-k) \times k$ matrix for which the equation $c \in C \leftrightarrow H_C c^T = 0$ takes place. For $x \in E^n$ denote by $A_i(x)$ the number of codewords of $C$ located at distance $i$ from $x$. The nonnegative integers $A_0^C, A_1^C, ..., A_n^C$, where $A_i^C = |\{c \in C / w(c) = i\}|$ are called weight spectra of code $C$. Let us denote by $W_C(x)$ the weight enumerator of code $C$: $W_C(x) = \sum_{i=0}^{n} A_i^C x^i$. A code $C$ will be called uniformly packed [3], if there are numbers $a_1, a_2, ..., a_{R_C}$ such that for all $x \in E^n$ the equation $\sum_{i=0}^{R_C} a_i A_i^C(x) = 1$ takes place. Denote by $K_j^n(x)$ the Kravchouk polynomial of degree $j$ [1,4], i.e.

$$K_i^n(x) = \sum_{j=0}^{i} (-1)^j \binom{n-x}{i-j} \binom{x}{j}, \text{ where } \binom{x}{j} = \frac{x(x-1)...(x-j+1)}{j!}.$$

Denote $L_C(x) = \sum_{i=0}^{R} a_i K_i^n(x)$. A code $C$ will be called quasi perfect [1,4], if $R_C = r_C + 1$. Many families of quasi perfect codes are known for the covering radius $\leq 4$ [5–10], but the general problem of existence of quasi-perfect codes by the given parameters is not completely solved yet [5]. When the geometrical interpretation of spherical covers is considered in the models of search of similarities, besides the perfect codes their other possible extensions can be considered and applied, such as quasi perfect codes or uniformly packed codes. The paper is organized as follows: in section 2 is brought definitions and coset weight structures of two error correcting primitive BCH codes and Golay codes, keeping in mind the fact that these are uniformly packed codes. Then in section 3 we consider the Elias algorithm for hash function obtained via these codes and get the formula representation of complexity of the algorithm using coset weight structure of mentioned codes.

## 2. Preliminaries.

*2.1. Coset Weight Distribution of Uniformly Packed Codes.* For a linear code $C$ we introduced the coset as the shift of the code. Later we need the coset weight distributions of two error-correcting BCH codes for lenght $n = 2^{2s+1} - 1$ and for the Golay code. As these codes can be considered as uniformly packed codes [3] we can find mentioned distributions by the method which brought in [3]:

***T h e o r e m 1.*** Let $C$ be uniformly packed code with parameters $a_0, a_1, .., a_n$. Then the polinomial $L_C(x)$ has $R$ distinct roots between 0 and $n$ [3].

Let us denote those roots by $\xi_1, ..., \xi_R$. Mention that if $C$ is uniformly packed code containing zero vector, then there exists a uniformly packed code with the same parameters and with the minimum weight $b$, where $0 \leq b \leq R$ which we denote by $C_b$. From the proof of the Theorem 1 follows:

***T h e o r e m 2.*** For the weight function of the uniformly packed code $C_b$ takes place the following equality [3]

$$W_{C_b}(x) = \frac{(1+x)^n}{\sum_{i=0}^{R} a_i \binom{n}{i}} + \sum_{i=1}^{R} B_{\xi_i}^b (1+x)^{n-\xi_i} (1+x)^{\xi_i}. \quad (1)$$

In (1) $B^b_{\xi_i}$'s are constants, which can be calculated from (1) by equalizing the corresponding coefficients in left and right sides and assuming that we know first $R$ coefficients of $W_{C_b}(x)$. In other words, to find the coefficients $B^b_{\xi_i}$'s we must solve the corresponding linear system of $R$ equations whith $R$ variables assuming that we know first $R$ values of weight spectra of the code $C_b$. From the Theorem 2 follows:

$$A_i^{C_b} = \frac{\binom{n}{i}}{\sum_{j=1}^{R} a_j \binom{n}{j}} + \sum_{j=1}^{R} B^b_{\xi_j} K_i^n(x_j). \tag{2}$$

Consequently to know the coset weight distributions of the uniformly packed code, we must calculate only first $R$ coset weights, which are $A_0^{C_b}, A_1^{C_b}, ..., A_{R-1}^{C_b}$.

*2.2 Two-Error-Correcting Primitive BCH Codes.* Let us denote the finite field of $q$ elements ($q$ is a power of a prime number) by $F_q$. We will consider finite fields with characterstic 2 [1]. Denote by $\alpha$ the primitive element of the fild $F_q$. Consider the set of formal polinomials $F_q[x]$ with coefficients from the field $F_q$. As it is known [1], the factor ring $R[x] = F_q[x]/(x^n - 1)$ is a ring of principal ideals, i.e. each ideal in $R[x]$ is principal. A $[n,k,d]R$ code $C$ will be called cyclic, if it is linear and from $c = (c_1, c_2, ..., c_n) \in C$ follows that $(c_n, c_1, ..., c_{n-1}) \in C$. We can correspond to each vector $(c_1, c_2, ...c_n)$ the polynomial $c_1 + c_2x + ... + c_nx^{n-1}$, so we can consider a code as a subset of $R[x]$. It is known [1], that each cyclic code is an ideal of $R[x]$ and consequently there is a unique monic polynomial $g(x)$ (generator polynomial) of minimum degree such that $\forall c \in C \exists f(x), c(x) = g(x)f(x)$, where the multiplication is taken in $R[x]$. Two-error-correcting primitive BCH codes are defined as cyclic codes for lenghts $n = 2^m - 1$ [1, 4]. The generator polynomal is $g(x) = scm\{M_\alpha(x), M_{\alpha^3}(x)\}$, where by $M_{\alpha^i}(x)$ is denoted the minimal polynomial of the element $\alpha^i$. It is known, that these codes have $2^m - 2m - 1$ and minimum distance 5 [1]. Also it is known that two-error-correcting primitive BCH codes are quasi-perfect codes [1, 11]. Weight distribution of these codes is calculated in [1, 12]. For odd $m$ two-error-correcting BCH codes are uniformly packed [3] with parameters $a_0 = a_1 = 1, a_2 = a_3 = \frac{6}{n-1}$. Roots of $L_{BC}(x)$ are $\xi_1 = \frac{n+1}{2} - \sqrt{\frac{n+1}{2}}$, $\xi_2 = \sqrt{\frac{n+1}{2}}$ and $\xi_3 = \frac{n+1}{2} + \sqrt{\frac{n+1}{2}}$. It is known that for odd $m$ there are four distinct weight distributions [3] and for even $m$ there are eight distinct weight distributions brought in [13].

*2.3 Golay Code.* First let us define the extended Golay code which has lenght 24. Let $A_{11}$ be the Hadmard matrix of Paley type [1], i.e.

$$A_{11} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Let $I_{11}$ be the identity matrix of size $11 \times 11$. The generator matrix of extended Golay code $[1,4]$ is the following

$$G_{\Gamma_{24}} = \begin{pmatrix} 1_{11}^T & I_{11} & 1_{11}^T & A_{11} \\ 0 & 0_{11} & 1 & 1_{11} \end{pmatrix},$$

where by $0_m$ and $1_m$ is denoted the all zero and one vectors respectively of lenght $m$. The Golay code $\Gamma_{23}$ is obtained by deleting the last coordinate from each codeword of extended Golay code. It is known $[1,4]$, that $\Gamma_{23}$ is a perfect three-error-correcting $[23,12,7]3$ code therefore we can consider it as uniformly packed code with parameters $a_0 = a_1 = a_2 = a_3 = 1$. The roots of $L_{\Gamma_{23}}(x)$ are $\xi_1 = 8$, $\xi_2 = 12$ and $\xi_3 = 16$.

**3. Elias Algorithm.** Let we have a subset (or a file) $F \in E^n$ and a query element $x \in E^n$. Let us consider the problem of finding the set of all "nearest neighbors" of $F$ to $x$. More precisely it is required to find the set $b(x,F) = \{y \in F/d(x,y) = c\}$, where $c = \min_{z \in F} d(x,z)$. To propose an algorithm for solving the problem of nearest neighbors in applied level, hash coding schemas are considered [14, 15]. We will brought a brief description of such schemes. Hash function is defined as a function $h: E^n \to V$, where $V = \{v_1,...,v_N\}$ is a finite set of $N$ elements [14]. In some cases it is possible that $u \neq v$, but $h(u) = h(v)$. Such situations are called collisions. The problem of collisions is solved by the technique called "chaining" [14]. The technicue is to keep $N$ distinct linked lists(or buckets) $L_i$ one for each possible hash value. For $i \in \{1,...,N\}$ denote by $B_i$ the set $\{x \in E^n/h(x) = v_i\}$. $B_i$'s are called blocks. The $i$-th list stores those vectors belonging to $F$, which have the same hash value, i.e. $L_i = \{x \in F/h(x) = i\}$ or in other way $L_i = B_i \bigcap F$. Hash coding scheme is called balanced, if $|B_i| = \frac{2^n}{N}$. The Elias algorithm [15] considers blocks $B_i$ ordering them by their distances at vector $x$. Mention that we must have an efficient method to find all blocks $B_{j_1},..,B_{j_{s(j)}}$ located at distance $j$ from $x$ if such blocks exist. After the step of ordering the algorithm examines the lists $L_{j_1},..,L_{j_{s(j)}}$ one after the other by increase of $j$. Let the best match distance is denoted by $\delta$. Due to $F \neq \oslash$ initialisation of $\delta$ will happen on some step. Now, if the current values obey $\delta < j$ algorithm stops the work. All blocks with higher distances than $\delta$ at $x$ do not need to be examined. In the reminder case $\delta \geq j$, examining nonempty list $L_{j_k}$ algorithm can change the best match distance $\delta$, also refreshing the current best match set, or the $\delta$ will remain unchanged and the current best match set will be updated. For balanced hash coding schemes it is proposed that the Elias algorithm may be optimal when the blocks $B_i$ are isoperimetric sets [15, 16]. By the complexity of algorithm we mean the average number of examined lists over all files and queries, supposing that each vector $z \in E^n$ can independently appear in $F$ with the same probability $p$. The pseudocode of the algorithm is brought below, where $n$ is the word lenghth, $N$ is the number of blocks.

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Elias algorithm
input $x$,$F$, comment: $F \neq \oslash$ integer $\delta = \infty$, comment: the current best match distance
set $S = \oslash$, comment: $S$ is the current set of vectors of $F$ located at distance $\delta$ from $x$
integer $j = -1$,
while($j < \delta$)
$\{$ $j{+}{+}$,
if($s(j) \neq 0$)
for(*integer* $i = 0, i < s(j), i{+}{+}$)
$\{$ if($L_{j_i} \neq \oslash$) comment: start examin the list $L_{j_i}$
if($\delta \leq d(x, L_{j_i})$) comment: $\delta$ is unchanged

$S = S\bigcup(O^n_\delta(x)\bigcap L_{j_i})$
else { comment: $\delta$ is changed
$S = O^n_\delta(x)\bigcap L_{j_i}$,
$\delta = d(x, L_{j_i})\}\}\}$
return $S$, comment: $s = b(x, f)$, $\delta = d(x, f)$

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Now suppose we have an $[n,k,d]R$ code $C$. We define a hash function $h : E^n \to C$ associated to the code $C$ in the following way:

$$h_C(x) = \{c_i / d(x, c_i) = d(x, C)\}. \tag{3}$$

As it follows from (3), $h_C(x)$ could be multivalued function becouse the blocks $B_i$ are spheres of radius $R$ and they can intersect. When the code $C$ is perfect the mentioned blocks do not intersect and their union covers unit cube. The formula for complexity of algorithm is brought below for the case corresponding to Golay code. But as perfect codes exist in very simple cases [1,2], we also consider hash functions associated to codes in some sense near to perfect codes. Such property have also the so called quasi-perfect codes [4–10]. Indeed the algorithm is proposed for balanced hash coding schemes, where different blocks $B_i$ do not intersect, but we will also consider the algorithm for the case of intersecting blocks. In this case when blocks intersect we create the list in a similar way to the basic case and then these lists are also intersecting. Repeated element bring some redundancy (in terms of memory). The formal expression of complexity of algorithm is then brought for the particular case of two-error-correcting primitive BCH code of length $2^m$–1 for odd $m$. To write a formula of complexity of the algorithm, for $x \in E^n$ let us consider the following Table:

| $x$ | $p_1$ $F_1$ | $p_2$ $F_2$ | … … | $p_{2^{2^n}}$ $F_{2^{2^n}}$ |
|---|---|---|---|---|
| $B_1$ | $a^x_{11}$ | $a^x_{12}$ | … | $a^x_{12^{2^n}}$ |
| $B_2$ | $a^x_{21}$ | $a^x_{22}$ | … | $a^x_{22^{2^n}}$ |
| … | … | … | … | … |
| $B_{2^k}$ | $a^x_{2^k1}$ | $a^x_{2^k2}$ | … | $a^x_{2^k2^{2^n}}$ |

In Table $F_1, ..., F_{2^{2^n}}$ are all subsets of vertexes of unit cube and each $F_i$ could be generated with the corresponding probability $p_i$. We will use the values $a^x_{ij}$ putting them in the cells corresponding to block $B_i$ and subset $F_j$, where

$$a^x_{ij} = \begin{cases} 1, & \text{if } B_i \text{ considered in case of the set } F_i \text{ and vertex } x, \\ 0, & \text{oterwise.} \end{cases}$$

As we mantion the complexity of algorithm will be represented as

$$\alpha(h_C) = \frac{1}{2^n} \sum_{x\in E^n} \sum_{1\le i\le 2^k} \sum_{1\le j\le 2^{2^n}} p_j a^x_{ij}. \tag{4}$$

Let us denote $\Phi_x(B_i) = \sum_{1\le j\le 2^{2^n}} p_j a^x_{ij}$. As we can see $\Phi_x(B_i)$ is the probability that the block $B_i$ will be considered by the algorithm when the vector $x$ is requested. Then

$$\alpha(h_C) = \frac{1}{2^n} \sum_{x\in E^n} \sum_{1\le i\le 2^k} \Phi_x(B_i).$$

It is easy to understand that for a fixed query $x$ the block $B_i$ will be examined, if the sphere $S^n_{d(x,B_i)-1}(x)$ does not contain any vector belonging to $F$. In that case all blocks $B_l$ such that

$d(x,B_l) \leq d(x,B_i) - 1$ will be examined. Let $j$ vary over all possible distances between vector $x$ and blocks $B_i$. Denote by $T_x(j)$ the number of blocks located at distance $\leq j$ from vector $x$ then

$$\alpha(h_C) = \frac{1}{2^n} \sum_{x \in E^n} \sum_{0 \leq j \leq n} T_x(j) V(j), \tag{5}$$

where $V(j)$ denotes the probability that the neares vector in $F$ is located at distance $j$ from $x$. Recall that [15] $V(j) = (1-(1-p)^{\binom{n}{j}})(1-p)^{\sum_{i=0}^{j-1}\binom{n}{i}}$. As $d(x,c_i) = w(x+c_i)$ then the number of vectors located at distance $i$ is equal to $A_i^{x+C}$. The sphere with centre $c_i$ and radius $R$ will be located at a distance $\leq j$ from vector $x$, if and only if $d(x,c_i) \leq j+R$. Therefore, $T_x(j) = \sum_{i=0}^{j+R} A_i^{x+C}$. Note that $A_i^{x+C} = 0$ when $i > n$.

As it is known the Golay code has four types of cosets [1] and each type can be obtained by some vector $e_i$ of weight $i$, $i \in \{0,1,2,3\}$. The number of cosets of minimum weight $i$ is equal to $\binom{23}{i}$, and each coset contain $2^{12}$ vectors. Therefore, we get the following:

*Proposition 1.* For the Golay code the complexity of Elias algorithm is:

$$\alpha(h_{\Gamma_{23}}) = \sum_{0 \leq j \leq 23} V(j) \sum_{i=0}^{j+3} \Big(\frac{1}{2^{11}} A_i^{e_0+\Gamma_{23}} + \frac{23}{2^{11}} A_i^{e_1+\Gamma_{23}} + \\ + \frac{253}{2^{11}} A_i^{e_2+\Gamma_{23}} + \frac{5819}{2^{11}} A_i^{e_3+\Gamma_{23}}\Big). \tag{6}$$

Proposition 1 gives the theoretical explanation of the experemental results, which are brought in [15]. As we mention for odd $m$ two-error correcting BCH codes has four types of coset. Keeping in maind this and calculating the number of each type from (5) we get:

*Proposition 2.* For the two-error-correcting BCH code the complexity of Elias algorithm is:

$$\alpha(h_{BC_m}) = \sum_{0 \leq j \leq 2^m-1} V(j) \sum_{i=0}^{j+3} \Big(\frac{1}{2^{2m}} A_i^{e_0+BC_m} + \frac{2^m-1}{2^{2m}} A_i^{e_1+BC_m} + \\ + \frac{(2^m-1)(2^{m-1}-1)}{2^{2m}} A_i^{e_2+BC_m} + \frac{2^{2m-1}+2^{m-1}-1}{2^{2m}} A_i^{e_3+BC_m}\Big). \tag{7}$$

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