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Mathematics

ASYMPTOTICS OF THE NORMING CONSTANTS OF THE STURM–LIOUVILLE PROBLEM

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We derive new asymptotic formulae for the norming constants of Sturm–Liouville problem, which generalize and make more precise previously known formulae, by taking into account the smooth dependence of norming constants on boundary conditions.

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1. Introduction and Statement of the Result.

Let $L(q,\alpha,\beta)$ denote the Sturm–Liouville problem

$$-y''+q(x)y=\mu y,\ x\in(0,\pi),\ \mu\in\mathbb{C}, \tag{1.1}$$

$$y(0)\cos\alpha+y'(0)\sin\alpha=0,\ \alpha\in(0,\pi], \tag{1.2}$$

$$y(\pi)\cos\beta+y'(\pi)\sin\beta=0,\ \beta\in[0,\pi), \tag{1.3}$$

where q is a real-valued, summable function on $[0,\pi]$ (we write $q\in L^1_{\mathbb{R}}[0,\pi]$). By $L(q,\alpha,\beta)$ we also denote the self-adjoint operator, generated by the problem (1.1)–(1.3) in Hilbert space $L^2[0,\pi]$ [1, 2]. It is well-known that the spectra of $L(q,\alpha,\beta)$ is discrete and consists of real, simple eigenvalues [1–3], which we denote by $\mu_n(q,\alpha,\beta)$, $n=0,1,2\ldots$, emphasizing the dependence of μ_n on q, α and β. For μ_n is proved the following asymptotic formula [4]

$$\mu_n(q,\alpha,\beta)=[n+\delta_n(\alpha,\beta)]^2+ \\ +\frac{1}{\pi}\int_0^\pi q(t)dt+r_n(q,\alpha,\beta), \tag{1.4}$$

where δ_n is the solution of the equation

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$$\delta_n(\alpha,\beta)=\frac{1}{\pi}\arccos\frac{\cos\alpha}{\sqrt{[n+\ \delta_n(\alpha,\beta)]^2\sin^2\alpha+\ \cos^2\alpha}}-$$

$$-\frac{1}{\pi}\arccos\frac{\cos\beta}{\sqrt{[n+\ \delta_n(\alpha,\beta)]^2\sin^2\beta+\ \cos^2\beta}}, \qquad (1.5)$$

and $r_n(q,\alpha,\beta)=o(1)$, when $n\to\infty$, uniformly in $\alpha,\beta\in[0,\pi]$ and q from any bounded subset of $L^1_{\mathbb{R}}[0,\pi]$ (we will write $q\in BL^1_{\mathbb{R}}[0,\pi]$). It follows from (1.5) (see [4] for details), that

$$\delta_n(\alpha,\beta)=\frac{\mathrm{ctg}\beta-\mathrm{ctg}\alpha}{\pi n}+O\left(1/n^2\right),\quad \alpha,\beta\in(0,\pi), \qquad (1.5a)$$

$$\delta_n(\pi,\beta)=\frac{1}{2}+\frac{\mathrm{ctg}\beta}{\pi\left(n+\frac{1}{2}\right)}+O\left(1/n^2\right)=\frac{1}{2}+O(1/n),\quad \beta\in(0,\pi), \qquad (1.5b)$$

$$\delta_n(\alpha,0)=\frac{1}{2}-\frac{\mathrm{ctg}\alpha}{\pi\left(n+\frac{1}{2}\right)}+O\left(1/n^2\right)=\frac{1}{2}+O(1/n),\quad \alpha\in(0,\pi), \qquad (1.5c)$$

$$\delta_n(\pi,0)=1. \qquad (1.5d)$$

Let $y=\varphi(x,\mu,\alpha,q)$ and $y=\psi(x,\mu,\beta,q)$ be the solutions of (1.1) with initial values

$$\varphi(0,\mu,\alpha,q)=\sin\alpha,\ \varphi'(0,\mu,\alpha,q)=-\cos\alpha,$$
$$\psi(\pi,\mu,\beta,q)=\sin\beta,\ \psi'(\pi,\mu,\beta,q)=-\cos\beta.$$

The eigenvalues μ_n of $L(q,\alpha,\beta)$ are the solutions of the equation

$$\Phi(\mu)=\varphi(\pi,\mu,\alpha,q)\cos\beta+\varphi'(\pi,\mu,\alpha,q)\sin\beta=$$
$$=-\left[\psi(0,\mu,\beta,q)\sin\alpha+\psi'(0,\mu,\beta,q)\cos\alpha\right]=0.$$

It is easy to see that for arbitrary $n=0,1,2,\ldots,$ $\varphi_n(x)\overset{def}{=}\varphi(x,\mu_n(q,\alpha,\beta),\alpha,\ q)$ and $\psi_n(x)\overset{def}{=}\psi(x,\mu_n(q,\alpha,\beta),\beta,q)$ are eigenfunctions, corresponding to the eigenvalue $\mu_n(q,\alpha,\beta)$. The squares of the L^2-norm of these eigenfunctions:

$$a_n(q,\alpha,\beta)\overset{def}{=}\int_0^\pi|\varphi_n(x)|^2dx,\ \ b_n(q,\alpha,\beta)\overset{def}{=}\int_0^\pi|\psi_n(x)|^2dx$$

are called the norming constants.

The main result of this paper is the following theorem:

Theorem. For norming constants a_n and b_n the following asymptotic formulae hold (when $n\to\infty$):

$$a_n(q,\alpha,\beta)=\frac{\pi}{2}\left[1+O\left(\frac{1}{n^2}\right)\right]\sin^2\alpha+\frac{\pi\cos^2\alpha}{2[n+\delta_n(\alpha,\beta)]^2}\left[1+O\left(\frac{1}{n^2}\right)\right], \qquad (1.6)$$

$$b_n(q,\alpha,\beta)=\frac{\pi}{2}\left[1+O\left(\frac{1}{n^2}\right)\right]\sin^2\beta+\frac{\pi\cos^2\beta}{2[n+\delta_n(\alpha,\beta)]^2}\left[1+O\left(\frac{1}{n^2}\right)\right], \qquad (1.7)$$

where the estimates of the remainders are uniform in $\alpha, \beta \in [0, \pi]$ and $q \in BL^1_{\mathbb{R}}[0, \pi]$.

In this paper we will prove (1.6). The proof of (1.7) is very similar (see the end of paper).

The dependence of norming constants on α and β (as far as we know) hasn't been investigated earlier. The dependence of spectral data (by spectral data here we understand the set of eigenvalues and the set of norming constants) on α and β is usually studied [1–9] in the following sense: the boundary conditions are separated into four cases:

1) $\sin\alpha \neq 0,\ \sin\beta \neq 0,\ $ i.e. $\alpha, \beta \in (0, \pi)$;

2) $\sin\alpha = 0,\ \sin\beta \neq 0,\ $ i.e. $\alpha = \pi,\ \beta \in (0, \pi)$;

3) $\sin\alpha \neq 0,\ \sin\beta = 0,\ $ i.e. $\alpha \in (0, \pi),\ \beta = 0$;

4) $\sin\alpha = 0,\ \sin\beta = 0,\ $ i.e. $\alpha = \pi,\ \beta = 0$,

and results are formulated separately for each case. For eigenvalues, formula (1.4) generalizes and unites four different formulae that were known earlier in four mentioned cases [4].

For norming constants the following is known as far.

In the case $\sin\alpha \neq 0$ it is known for smooth q that

$$\frac{a_n(q, \alpha, \beta)}{\sin^2\alpha} = \frac{\pi}{2} + O\left(\frac{1}{n^2}\right). \tag{1.8}$$

For absolutely continuous q (we will write $q \in AC[0, \pi]$) the proof of (1.8) can be found in [2]. In [10], under the condition $q(x) = \dfrac{dF(x)}{dx}$ (almost everywhere), where F is a function of bounded variation (we will write $F \in BV[0, \pi]$), author asserts that

$$\frac{a_n(q, \alpha, \beta)}{\sin^2\alpha} = \frac{\pi}{2} + \alpha_n, \tag{1.9}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is characterized by the condition that the function $f(x) \overset{def}{=} \sum_{n=0}^{\infty} \alpha_n \cos nx$ has a bounded variation on [0,π], i.e. $f \in BV[0, \pi]$.

In [9], for $q \in L^2_{\mathbb{R}}[0, \pi]$, it was proved that

$$\frac{a_n(q, \alpha, \beta)}{\sin^2\alpha} = \frac{\pi}{2} + \frac{\kappa_n}{n} \tag{1.10}$$

where $\{\kappa_n\}_{n=0}^{\infty} \in l^2$ (i.e. $\sum_{n=0}^{\infty} |\kappa_n|^2 < \infty$). It is easy to see that our result (1.6) is more general, since it shows that (for $\sin\alpha \neq 0$) the estimate (1.8) holds for arbitrary $q \in L^1_{\mathbb{R}}[0, \pi]$.

It is also important to note that norming constants $a_n(q, \alpha, \beta)$ are analytic functions on α and β. It easily follows from formulae (3.1), (3.2) and (3.4) below and from the result [4], that $\lambda_n(q,\alpha,\beta)$ $(\lambda_n^2(q,\alpha,\beta)=\mu_n(q,\alpha,\beta))$ depend analytically on α and β.

In the case $\sin\alpha=0$, $\sin\beta\neq 0$ it is known for smooth q (for $q\in AC[0,\pi]$ the proof of (1.11) can be found in [2]), that

$$a_n(q,\pi,\beta)=\frac{\pi}{2(n+1/2)^2}\left[1+O\left(\frac{1}{n^2}\right)\right]. \tag{1.11}$$

Since $\delta_n(\pi,\beta)=\frac{1}{2}+O\left(\frac{1}{n}\right)$ (1.5b), it is easy to see that (1.11) follows from (1.6). Besides, we see that (1.6) smoothly turn into (1.11) when $\alpha\to\pi$.

In the case $\sin\alpha=0$, $\sin\beta=0$ the following result can be found in [2] for $q\in AC[0,\pi]$:

$$a_n(q,\pi,0)=\frac{\pi}{2n^2}\left[1+O\left(\frac{1}{n^2}\right)\right].$$

We think that it is more correct to write this result in the form (note that $\delta_n(\pi,0)=1$)

$$a_n(q,\pi,0)=\frac{\pi}{2(n+1)^2}\left[1+O\left(\frac{1}{n^2}\right)\right] \tag{1.12}$$

to keep the beginning of the enumeration of eigenvalues and norming constants start from 0, but not from 1, as is in [2].

It is easy to see that our result (1.6) covers all previous results (1.8)–(1.12) and is more precise. Our proof of the Theorem is based on the detailed study of the dependence of eigenfunctions φ_n and ψ_n on parameters α and β. We will present it in the Section 3. But first we need to prove some properties of solutions of Eq. (1.1).

2. Asymptotics of the Solutions. Let $q\in L^1_{\mathbb{C}}[0,\pi]$, i.e. q is a complex-valued, summable function on $[0,\pi]$, and let us denote by $y_i(x,\lambda)$, $i=1,2,3,4$, the solutions of the equation

$$-y''+q(x)y=\lambda^2 y, \tag{2.1}$$

satisfying the initial conditions

$$\begin{gathered} y_1(0,\lambda)=1,\quad y_2(0,\lambda)=0,\quad y_3(\pi,\lambda)=1,\quad y_4(\pi,\lambda)=0,\\ y_1'(0,\lambda)=0,\quad y_2'(0,\lambda)=1,\quad y_3'(\pi,\lambda)=0,\quad y_4'(\pi,\lambda)=1. \end{gathered} \tag{2.2}$$

Let recall that by a solution of (2.1) (which is the same as (1.1)) we understand the function y, s.t. $y,y'\in AC[0,\pi]$ and which satisfies (2.1) almost everywhere [1].

The solutions y_1 and y_2 (as well as the second pair y_3 and y_4) form a fundamental system of solutions of Eq. (1.1), i.e. any solution y of (1.1) can be represented in the form

$$y(x)=y(0)y_1(x,\lambda)+y'(0)y_2(x,\lambda)=y(\pi)y_3(x,\lambda)+y'(\pi)y_4(x,\lambda). \tag{2.3}$$

The existence and uniqueness of solutions $y_i,\ i=1,2,3,4$ (under the condition $q\in L^1_{\mathbb{C}}[0,\pi]$) were investigated in [1, 11–14]. The following lemma in some sense extends the results of the mentioned papers related to asymptotics (when $|\lambda|\to\infty$) of solutions $y_i,\ i=1,2,3,4$.

Lemma. Let $q\in L^1_{\mathbb{C}}[0,\pi]$. Then for solutions $y_i,\ i=1,2,3,4$, the following representations hold (when $|\lambda|\geq 1$:)

$$y_1(x,\lambda)=\cos\lambda x+\frac{1}{2\lambda}a(x,\lambda), \tag{2.4}$$

$$y_2(x,\lambda)=\frac{\sin\lambda x}{\lambda}-\frac{1}{2\lambda^2}b(x,\lambda), \tag{2.5}$$

$$y_3(x,\lambda)=\cos\lambda(\pi-x)+\frac{1}{2\lambda}c(x,\lambda), \tag{2.6}$$

$$y_4(x,\lambda)=\frac{\sin\lambda(\pi-x)}{\lambda}-\frac{1}{2\lambda^2}d(x,\lambda), \tag{2.7}$$

where a, b, c, d are twice differentiable in x and entire functions with respect to λ, and have the form

$$a(x,\lambda)=\sin\lambda x\int_0^x q(t)dt+\int_0^x q(t)\sin\lambda(x-2t)dt+R_1(x,\lambda,q), \tag{2.8}$$

$$b(x,\lambda)=\cos\lambda x\int_0^x q(t)dt-\int_0^x q(t)\cos\lambda(x-2t)dt+R_2(x,\lambda,q), \tag{2.9}$$

$$c(x,\lambda)=\sin\lambda(\pi-x)\int_x^\pi q(t)dt+\int_x^\pi q(t)\sin\lambda(2t-\pi-x)dt+R_3(x,\lambda,q), \tag{2.10}$$

$$d(x,\lambda)=\cos\lambda(\pi-x)\int_x^\pi q(t)dt+\int_x^\pi q(t)\cos\lambda(\pi+x-2t)dt+R_4(x,\lambda,q), \tag{2.11}$$

and $R_i,\ i=1,2,3,4$, satisfy the estimates (when $|\lambda|\geq 1$):

$$R_1(x,\lambda,q),\ R_2(x,\lambda,q)=O\left(\frac{e^{|\mathrm{Im}\lambda|x}}{|\lambda|}\right), \tag{2.12}$$

$$R_3(x,\lambda,q),\ \ R_4(x,\lambda,q)=O\left(\frac{e^{|\mathrm{Im}\lambda|(\pi-x)}}{|\lambda|}\right), \tag{2.13}$$

Proof. In [14] we have proved that $y_2(x,\lambda)$ we can be obtained in the form of series

$$y_2(x,\lambda,q)=\sum_{k=0}^{\infty}S_k(x,\lambda,q),$$

which converge to $y_2(x,\lambda,q)$ uniformly on bounded subsets of the set $[0,\pi]\times\mathbb{C}\times\times L^1_{\mathbb{C}}[0,\pi]$, and where $S_0(x,\lambda,q)=\dfrac{\sin\lambda x}{\lambda}$,

$$S_k(x,\lambda,q)=\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)S_{k-1}(t,\lambda,q)\,dt,\quad k=1,2,\dots$$

For S_k we have the estimate (when $|\lambda|\geq 1$)

$$|S_k(x,\lambda,q)|\leq \frac{e^{|\mathrm{Im}\lambda|x}}{|\lambda|^{k+1}}\cdot\frac{\sigma_0^k(x)}{k!},\ \ k=0,1,2,\dots, \tag{2.14}$$

where $\sigma_0(x)\equiv\int_0^x q(t)dt$ [14]. To prove (2.5), (2.9) and the estimate (2.12), we write S_1 in the form

$$S_1(x,\lambda,q)=\int_0^x \frac{\sin\lambda(x-t)}{\lambda}\cdot\frac{\sin\lambda t}{\lambda}q(t)\,dt=$$
$$=\frac{1}{2\lambda^2}\int_0^x[\cos\lambda(x-2t)-\cos\lambda x]q(t)\,dt=-\frac{\cos\lambda x}{2\lambda^2}\int_0^x q(t)\,dt+$$
$$+\frac{1}{2\lambda^2}\int_0^x\cos\lambda(x-2t)q(t)\,dt,$$

and we note that

$$S_k{}'(x,\lambda,q)=\int_0^x\cos\lambda(x-t)q(t)S_{k-1}(t,\lambda,q)\,dt,\ k=1,2,\dots$$

This implies that

$$S_k{}'\in AC[0,\pi].$$

By writing $y_2(x,\lambda,q)=S_0+S_1+\sum_{k=2}^{\infty}S_k(x,\lambda,q)$, we obtain

$$y_2(x,\lambda,q)=\frac{\sin\lambda x}{\lambda}-\frac{1}{2\lambda^2}b(x,\lambda),$$

where $-\frac{1}{2\lambda^2}b(x,\lambda)=\sum_{k=1}^{\infty}S_k(x,\lambda,q)$ and, therefore, $b(x,\lambda)$ has the form (2.9), where $R_2(x,\lambda)=-2\lambda^2\sum_{k=2}^{\infty}S_k(x,\lambda)$. Now from estimate (2.14) we obtain that

$$\sum_{k=2}^{\infty}|S_k(x,\lambda,q)|\leq\sum_{k=2}^{\infty}\frac{e^{|\mathrm{Im}\lambda|x}}{|\lambda|^{k+1}}\cdot\frac{\sigma_0^k(x)}{k!}=\frac{e^{|\mathrm{Im}\lambda|x}\sigma_0^2(x)}{|\lambda|^3}\sum_{k=2}^{\infty}\frac{\sigma_0^{k-2}(x)}{|\lambda|^{k-2}k!}<$$
$$<\frac{e^{|\mathrm{Im}\lambda|x}\sigma_0^2(x)}{|\lambda|^3}\sum_{k=2}^{\infty}\frac{\sigma_0^{k-2}(x)}{|\lambda|^{k-2}(k-2)!}=\frac{e^{|\mathrm{Im}\lambda|x}\sigma_0^2(x)}{|\lambda|^3}\sum_{n=0}^{\infty}\frac{\sigma_0^n(x)}{|\lambda|^n n!}=$$
$$=\frac{e^{|\mathrm{Im}\lambda|x}\sigma_0^2(x)}{|\lambda|^3}e^{\frac{\sigma_0(x)}{|\lambda|}}=\frac{\sigma_0^2(x)}{|\lambda|^3}e^{|\mathrm{Im}\lambda|x+\frac{\sigma_0(x)}{|\lambda|}}.$$

This implies (2.12) for $|\lambda|\geq 1$. Since $y_2{}'\in AC[0,\pi]$ and $S_0(x,\lambda)=\frac{\sin\lambda x}{\lambda}$, we obtain, that $-\frac{1}{2\lambda^2}b(x,\lambda)=y_2-S_0$ is also a twice differentiable function (more precisely

$b' \in AC[0,\pi]$). Assertions on y_1, y_3, y_4 can be proved similarly. □

3. Proof of the Theorem. According to (2.3), the solution $\varphi(x,\mu,\alpha,q)$ which we will denote by $\varphi\left(x,\lambda^2,\alpha\right)$ for bravity, has the form

$$\varphi\left(x,\lambda^2,\alpha\right) = y_1(x,\lambda)\sin\alpha - y_2(x,\lambda)\cos\alpha, \tag{3.1}$$

and, according to (2.4) and (2.5),

$$\varphi\left(x,\lambda^2,\alpha\right) = \left[\cos\lambda x + \frac{1}{2\lambda}a(x,\lambda)\right]\sin\alpha - \left[\frac{\sin\lambda x}{\lambda} - \frac{1}{2\lambda^2}b(x,\lambda)\right]\cos\alpha.$$

Taking the squares of both sides of equation above, we obtain:

$$\varphi^2\left(x,\lambda^2,\alpha\right) = \cos^2\lambda x\sin^2\alpha + \frac{1}{\lambda}\left[a(x,\lambda)\cos\alpha + \frac{\alpha^2(x,\lambda)}{4\lambda}\right]\sin^2\alpha -$$

$$-\frac{2}{\lambda}\left[\cos\lambda x\sin\lambda x - \frac{b(x,\lambda)\cos\lambda x}{2\lambda} + \frac{a(x,\lambda)\sin\lambda x}{2\lambda} - \frac{a(x,\lambda)b(x,\lambda)}{4\lambda^2}\right]\times \tag{3.2}$$

$$\times\sin\alpha\cos\alpha + \frac{\sin^2\lambda x}{\lambda^2}\cos^2\alpha + \left(\frac{b^2(x,\lambda)}{4\lambda^4} - \frac{b(x,\lambda)\sin\lambda x}{\lambda^3}\right)\cos^2\alpha.$$

Taking into account the formulae $\cos^2\lambda x = \frac{1}{2}(1+\cos 2\lambda x)$ and $\sin^2\lambda x = = \frac{1}{2}(1-\cos 2\lambda x)$, we obtain from (3.2):

$$\int_0^\pi \varphi^2\left(x,\lambda^2,\alpha\right)dx = \frac{\pi}{2}\sin^2\alpha + \frac{\sin 2\lambda\pi}{4\lambda}\sin^2\alpha +$$

$$+\frac{1}{\lambda}\left[\int_0^\pi a(x,\lambda)\cos\lambda x\,dx + \frac{1}{4\lambda}\int_0^\pi a^2(x,\lambda)\,dx\right]\sin^2\alpha - \frac{\sin^2\lambda\pi}{\lambda^2}\sin\alpha\cos\alpha +$$

$$+\frac{1}{\lambda^2}\left[\int_0^\pi b(x,\lambda)\cos\lambda x\,dx - \int_0^\pi a(x,\lambda)\sin\lambda x\,dx\right]\sin\alpha\cos\alpha + \tag{3.3}$$

$$+\frac{\sin\alpha\cos\alpha}{2\lambda}\int_0^\pi a(x,\lambda)b(x,\lambda)\,dx + \frac{\pi}{2\lambda^2}\cos^2\alpha - \frac{\sin 2\lambda\pi}{4\lambda^3}\cos^2\alpha -$$

$$-\frac{1}{\lambda^3}\left[\int_0^\pi b(x,\lambda)\sin\lambda x dx - \frac{1}{\lambda}\int_0^\pi b^2(x,\lambda)\,dx\right]\cos^2\alpha.$$

The asymptotic formula (1.6) can be obtained from (3.3) by substituting $\lambda = \lambda_n(q,\alpha,\beta) = \sqrt{\mu_n(q,\alpha,\beta)}$. Now we estimate the right-hand side of (3.3) for $\lambda = \lambda_n$. It can be easily deduced from (1.4) that for $\lambda_n = \sqrt{\mu_n}$ we have the following asymptotic formula $\left([q] = \frac{1}{\pi}\int_0^\pi q(t)\,dt\right)$:

$$\lambda_n(q,\alpha,\beta) = n + \delta_n(\alpha,\beta) + \frac{[q]}{2[n+\delta_n(\alpha,\beta)]} + l_n,$$

where $l_n = l_n(q,\alpha,\beta) = o\left(\frac{1}{n}\right)$ uniform in $\alpha,\ \beta \in [0,\pi]$ and $q \in BL^1_{\mathbb{R}}[0,\pi]$.

It follows from (1.5a)–(1.5d) that $\sin 2\pi\delta_n(\alpha,\beta) = O\left(\frac{1}{n}\right)$ for all $(\alpha,\beta) \in \in (0,\pi] \times [0,\pi)$ and also that

$$\sin 2\pi\lambda_n(q,\alpha,\beta) = O\left(\frac{1}{n}\right),\ \cos 2\pi\lambda_n(q,\alpha,\beta) = 1 - O\left(\frac{1}{n^2}\right)$$

also for all $(\alpha,\beta) \in (0,\pi] \times [0,\pi)$.

Thus, the second term $\frac{\sin 2\pi\lambda_n}{\lambda_n}\sin^2\alpha$ in (3.3) has the form $O\left(\frac{1}{n^2}\right)\sin^2\alpha$. Since functions $a(x,\lambda_n)$, $a^2(x,\lambda_n)$, $b(x,\lambda_n)$, $a(x,\lambda_n)b(x,\lambda_n)$, $b^2(x,\lambda_n)$ have absolutely continuous derivatives, their Fourier coefficients are at least $O\left(\frac{1}{\lambda_n}\right) = = O\left(\frac{1}{n}\right)$ uniformly for $(\alpha,\beta) \in (0,\pi] \times [0,\pi)$ and $q \in BL^1_{\mathbb{R}}[0,\pi]$.

Therefore, the third and fourth terms in (3.3) (for $\lambda = \lambda_n$) ("coefficients" of $\sin^2\alpha$) have the orders correspondingly $\frac{1}{\lambda_n}O\left(\frac{1}{n}\right) = O\left(\frac{1}{n^2}\right)$ and $\frac{1}{\lambda_n^2} = O\left(\frac{1}{n^2}\right)$.

The fifth term is $-\frac{\sin^2\pi\lambda_n}{\lambda_n^2}\sin\alpha\cos\alpha = \frac{-(1-\cos 2\pi\lambda_n)}{2\lambda_n^2}\sin\alpha\cos\alpha = = O\left(\frac{1}{n^4}\right)\sin\alpha\cos\alpha$.

The sixth, seventh, eighth terms (the coefficients of $\sin\alpha\cos\alpha$) have the order $\frac{1}{\lambda_n^2}O\left(\frac{1}{\lambda_n}\right) = O\left(\frac{1}{n^3}\right)$.

The last three terms, the "coefficients" of $\cos^2\alpha$ (except $\frac{\pi}{2\lambda_n^2}\cos^2\alpha$), have the order $\frac{1}{\lambda_n^4} = O\left(\frac{1}{n^4}\right)$. Therefore, the last four terms can be written in the form $\frac{\pi}{2\lambda_n^2}\left[1 + O\left(\frac{1}{n^2}\right)\right]\cos^2\alpha$.

Let us note that from $\mu_n = \lambda_n^2 = (n+\delta_n)^2 + [q] + r_n$ it follows that $\frac{1}{\lambda_n^2} = \frac{1}{[n+\delta_n]^2} + \eta_n$, where $\eta_n = \frac{1}{[n+\delta_n]^2 + [q] + r_n} - \frac{1}{[n+\delta_n]^2} = O\left(\frac{1}{n^4}\right)$.
Thus, we conclude that the expression (3.3) in $\lambda = \lambda_n$ have the form

$$a_n(q,\alpha,\beta) = \frac{\pi}{2}\left[1 + O\left(\frac{1}{n^2}\right)\right]\sin^2\alpha + O\left(\frac{1}{n^3}\right)\sin\alpha\cos\alpha + $$
$$+ \frac{\pi}{2[n+\delta_n(\alpha,\beta)]^2}\left[1 + O\left(\frac{1}{n^2}\right)\right]\cos^2\alpha. \qquad (3.4)$$

If $\sin\alpha \neq 0$, then the term $O\left(\frac{1}{n^3}\right)\sin\alpha\cos\alpha$ can be included into the term $O\left(\frac{1}{n^2}\right)\sin^2\alpha$, and if $\sin\alpha = 0$, then these terms are absent. Thus, finally, (3.4) we can write in the form (1.6). Formula (1.7) for b_n can be obtained similarly, if we represent $\psi(x,\mu,\beta,q) = y_3(x,\lambda)\sin\beta - y_4(x,\lambda)\cos\beta$ and use formulae (2.6), (2.7) and estimates from Lemma. We can obtain (1.7) also in another way: we can use the representation $\psi(x,\mu,\beta,q) = \varphi(\pi - x,\mu,\pi-\beta,q*)$, where $q* = q(\pi - x)$, which is well-known (see, e.g., [5]) and which is easy to verify by direct substitution. □

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