LOCALIZED BENDING VIBRATIONS
OF PIEZOCERAMIC TRANSVERSE POLARIZED PLATE

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Problem of the piezoceramic plate polarized along the normal of the middle plane of the plate is solved, based on the assumptions of the hypothesis of Kirchhoff, taking into account the components characterizing the electric field. The equations of planar and bending vibrations are obtained. Localized bending vibrations are considered, and the effect of the electric field on the frequency of localized vibrations is investigated.

MSC2010: 74H45.

Keywords: Kirchhoff’s hypothesis, natural frequencies, piezocrystal.

Introduction. The problems of vibrations of piezoceramic plates were studied in [1–6]. In [2, 3] bending vibrations of piezoceramic plate were studied, based on the assumptions of Kirchhoff’s hypothesis. In [4] the problem of bending vibrations of piezoceramic plate of class 6 mm was investigated polarized along the normal of the middle plane of the plate. The components of plate displacement are presented, taking into account the assumptions of refined theory and with allowance for the components characterizing the electric field.

In [5] the piezoceramic plate was considered, where the components of displacement are presented based on the theory of S. Ambartsumian.

In present paper localized bending vibrations of piezoceramic plate are considered, on the basis of Kirchhoff’s hypothesis, taking into account additional components characterizing the electric field.

Problem Setting. Consider piezoceramic plate of constant thickness $2h$ and polarized along the normal of the middle plane of the plate. The plate is located in the Cartesian coordinate system, so, that its middle plane coincides with the plane $XOY$ and the plate occupies the region $0 \leq x \leq a$, $0 \leq x \leq \infty$, $-h \leq z \leq h$.  

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The equations of state of an elastic body are written in the form [1]:

\[
\begin{align*}
\varepsilon_{11} &= \frac{(\sigma_{11} - v\sigma_{22})}{E} - \frac{\nu'}{E} \frac{\sigma_{33}}{E} + d_{31}E_3, \\
\varepsilon_{22} &= \frac{(\sigma_{22} - v\sigma_{11})}{E} - \frac{\nu'}{E} \frac{\sigma_{33}}{E} + d_{31}E_3, \\
\varepsilon_{33} &= \frac{\nu'}{E} (\sigma_{11} + \sigma_{22}) + \frac{1}{E} \sigma_{33} + d_{33}E_3, \\
\varepsilon_{12} &= \frac{2(1 + \nu)}{E} \sigma_{12}, \\
\varepsilon_{13} &= \frac{1}{G} \sigma_{13} + d_{15}E_1, \\
\varepsilon_{23} &= \frac{1}{G} \sigma_{23} + d_{15}E_2,
\end{align*}
\]

(1)

Here, \(\varepsilon_{ij}\) are components of the strain tensor; \(E\) and \(\nu\) are Young's modulus and Poisson's ratio, respectively; \(d_{ij}\) are piezoelectric coefficients; \(\sigma_{ij}\) are components of stress tensor; \(\nu'\) is the Poisson's ratio; \(G\) is the shear modulus; and \(E', \nu'\) are the effective Young's modulus and effective Poisson's ratio, respectively.

Neglecting the piezoelectric effect, the equations of state are written as:

\[
\begin{align*}
\sigma_{11} &= \frac{E}{1 - \nu^2} (\varepsilon_{11} + \nu \varepsilon_{22}) - \frac{E}{1 - \nu} d_{31}E_3, \\
\sigma_{22} &= \frac{E}{1 - \nu^2} (\varepsilon_{22} + \nu \varepsilon_{11}) - \frac{E}{1 - \nu} d_{31}E_3, \\
\sigma_{12} &= \frac{E}{2(1 + \nu)} \varepsilon_{12}, \\
D_1 &= \varepsilon_{11} E_1 + d_{15} \sigma_{13}, \\
D_2 &= \varepsilon_{11} E_2 + d_{15} \sigma_{23}, \\
D_3 &= \varepsilon_{33} E_3 + d_{13} (\sigma_{11} + \sigma_{22}) + d_{33} \sigma_{33},
\end{align*}
\]

(2)

where \(\varepsilon_{ij}\) are strain coefficients at zero mechanical stresses; \(d_{ij}\) are piezoelectric constants; \(E\) is the modulus of elasticity; \(\sigma_{ij}\) are components of stress tensor; \(\varepsilon_{ij}\) are components of strain tensor.

According to the assumption of Kirchhoff’s hypothesis from [1] and [2], and neglecting \(\sigma_{13}, \sigma_{23}, \sigma_{33}\) the equations of state are written as:

\[
\begin{align*}
\sigma_{11} &= \frac{E}{1 - \nu^2} (\varepsilon_{11} + \nu \varepsilon_{22}) - \frac{E}{1 - \nu} d_{31}E_3, \\
\sigma_{22} &= \frac{E}{1 - \nu^2} (\varepsilon_{22} + \nu \varepsilon_{11}) - \frac{E}{1 - \nu} d_{31}E_3, \\
\sigma_{12} &= \frac{E}{2(1 + \nu)} \varepsilon_{12}, \\
D_1 &= \varepsilon_{11} E_1 + d_{15} \sigma_{13}, \\
D_2 &= \varepsilon_{11} E_2 + d_{15} \sigma_{23}, \\
D_3 &= \varepsilon_{33} E_3 + d_{13} (\sigma_{11} + \sigma_{22}).
\end{align*}
\]

(3)

Equations of electrodynamics for piezomedium have the following view in the electrostatic approach:

\[
\begin{align*}
\text{div } \vec{D} &= 0, \\
\text{rot } \vec{E} &= \vec{0} \quad (\vec{E} = -\text{grad } \varphi).
\end{align*}
\]

(5)

For displacement component, at any point of the plate, according to the assumptions of Kirchhoff’s hypothesis, we have:

\[
\begin{align*}
\begin{array}{ccc}
u_1 &= u - z \frac{\partial w}{\partial x} + d_{15} \int_0^z E_1 d\xi, & \nu_2 &= v - z \frac{\partial w}{\partial x} + d_{15} \int_0^z E_2 d\xi, & \nu_3 &= w,
\end{array}
\end{align*}
\]

(6)

here \(u, v, w\) are displacement of the middle plane of the plate.

For the problem of bending vibration of piezoceramic plate, displacements of the middle plane of the plate are obtained, according to the Kirchhoff’s hypothesis assumption, where the integrals members are not taken into account [2][3].

In [5] displacement components are considered taking into account the refined theory of S. Ambartsumian, and in special case displacements of the middle plane of the plate are obtained according to the Kirchhoff’s hypothesis, where the components are not taken into account characterizing electric field.

Stress components and components of the electric induction vector, taking into account [6], are written by the following way:
where the following notes are accepted:

\[ \sigma_{11} = \frac{E}{1-v^2} \left( \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} \right) - \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right) - d_{15} \int_0^r \left( \frac{\partial^2 \varphi}{\partial x^2} + v \frac{\partial^2 \varphi}{\partial y^2} \right) d\xi \]

\[ \sigma_{22} = \frac{E}{1-v^2} \left( \frac{\partial v}{\partial y} + v \frac{\partial u}{\partial x} \right) - \left( \frac{\partial^2 w}{\partial y^2} + v \frac{\partial^2 w}{\partial x^2} \right) - d_{15} \int_0^r \left( \frac{\partial^2 \varphi}{\partial y^2} + v \frac{\partial^2 \varphi}{\partial x^2} \right) d\xi \]

\[ \sigma_{12} = \frac{E}{2(1+v)} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial^2 w}{\partial x \partial y} - 2d_{15} \int_0^r \frac{\partial^2 \varphi}{\partial x \partial y} d\xi \right). \]

\[ D_1 = -\epsilon_{11} \frac{\partial \varphi}{\partial x}, \quad D_2 = -\epsilon_{11} \frac{\partial \varphi}{\partial y}, \quad D_3 = -\epsilon_{33} \frac{\partial \varphi}{\partial z} \]

\[ \frac{d^3 E}{d^3 z} \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{d^3 E}{d^3 z} \frac{\partial^2 \varphi}{\partial y \partial x} = \frac{d^3 E}{d^3 z} \frac{\partial^2 \varphi}{\partial x \partial y}. \]

Motion equation of piezoceramic body:

\[ \frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}. \]
Integrating the motion equations \( \frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial y} + \sigma_{13}^{h} \big|_{-h} = 2ph \frac{\partial^2 u}{\partial t^2} \), \( \frac{\partial T_2}{\partial x} + \frac{\partial S}{\partial y} + \sigma_{23}^{h} \big|_{-h} = 2ph \frac{\partial^2 v}{\partial t^2} \), \( \frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial y} + \sigma_{33}^{h} \big|_{-h} = 2ph \frac{\partial^2 w}{\partial t^2} \) (13) on \( z \) by the limits from \(-h\) to \( h\), we will get:

\[
\frac{\partial M_1}{\partial x} + \frac{\partial H}{\partial y} = N_1, \quad \frac{\partial M_2}{\partial x} = N_2.
\]

(14)

Substituting the internal forces values and the moments in the first two equations \((13), (14)\), we have:

\[
2hE \left( \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 v}{\partial x \partial y} + hE \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + Ed_{15} \frac{\partial}{\partial x} \left( \phi_{1h}^{h} \right) \right) - Ed_{15} \left( \frac{\partial^3}{\partial x \partial y^2} + \frac{\partial^3}{\partial x \partial y^2} \right) (l_1 + m_1) + \sigma_{13}^{h} \big|_{-h} = 2ph \frac{\partial^2 u}{\partial t^2} - \rho d_{15} \frac{\partial^3}{\partial x \partial y^2} (l_1 + m_1),
\]

(15)

\[
2hE \left( \frac{\partial^2 u}{\partial y^2} + v \frac{\partial^2 u}{\partial x \partial y} + hE \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} \right) + Ed_{13} \frac{\partial}{\partial x} \left( \phi_{1h}^{h} \right) \right) - Ed_{15} \left( \frac{\partial^3}{\partial x \partial y^2} + \frac{\partial^3}{\partial x \partial y^2} \right) (l_1 + m_1) + \sigma_{23}^{h} \big|_{-h} = 2ph \frac{\partial^2 v}{\partial t^2} - \rho d_{15} \frac{\partial^3}{\partial y \partial t^2} (l_1 + m_1),
\]

(16)

The system (13) represents the equations of planar vibrations of a piezoceramic plate. Substituting the last equation from (13) in (16), we obtain the equation of bending vibrations:

\[
D \Delta^2 w + 2ph \frac{\partial^2 w}{\partial t^2} - Ed_{13} \left( \frac{\partial}{\partial x} \left( \phi_{1h}^{h} \right) - \frac{1}{h} \int_{-h}^{h} \phi dz \right) + Ed_{15} \frac{\partial}{\partial x} \Delta^2 (l_2 + m_2) - \sigma_{33}^{h} \big|_{-h} = 0,
\]

(17)

where \( D = \frac{2h^3 E}{3(1 - v^2)} \).

Averaging the equation of electrodynamics, the following relations are obtained:

\[
- (\epsilon_{11} + \frac{d_{13} d_{15} E}{1 - v}) \Delta \phi - (\epsilon_{33} + \frac{d_{13} d_{23} E}{1 - v}) \frac{\partial^2 \phi}{\partial z^2} \frac{d_{13} E}{1 - v} \Delta w = 0.
\]

(18)

Let’s consider the equation of bending vibrations. On the surface faces of the plate \( z = \pm h \) has the following condition:

\[
\sigma_{13} = 0, \quad \sigma_{23} = 0, \quad \sigma_{33} = 0, \quad \sigma = 0.
\]

(19)
Then the equations of bending vibrations and for the potential will be written:

\[ D\Delta^2 w + 2\rho h \frac{\partial^2 w}{\partial t^2} + Eh\frac{d}{1-v} \Delta \left( \frac{1}{h} \int_{-h}^{h} \varphi dz \right) + \frac{Eh_{13}}{1-v} \Delta^2 (I_2 + m_2) = 0, \]

\[ \Delta \varphi + a_1 \frac{\partial^2 \varphi}{\partial z^2} = -a_2 \Delta w, \]  

(20)

where the following notes are accepted:

\[ a_1 = \frac{\varepsilon_{31}}{11} \frac{1}{1-v} + d_{13}E, \quad a_2 = \frac{d_{13}E}{11} (1-v) + d_{13}d_{15}E. \]

In this case the equations of bending vibrations are separated from the equations of planar vibrations.

Suppose that the following boundary conditions are given at the edges of the plate and in the limit \( y \to \infty \):

- at \( x = 0 \), \( v = 0 \), \( M_1 = 0 \), \( \varphi = 0 \);
- at \( x = a \), \( v = 0 \), \( M_1 = 0 \), \( \varphi = 0 \);
- at \( y = 0 \), \( M_2 = 0 \), \( \frac{\partial M_1}{\partial y} + 2 \frac{\partial H}{\partial y} = 0 \), \( \varphi = 0 \);
- at \( y \to \infty \), \( v = 0 \), \( \varphi = 0 \).

(21)

The solution of the Eqs. (20) of the system we will seek in the form

\[ w(x,y,t) = w_0(y) \sin \alpha_n x \exp(i\omega t), \quad \varphi(x,y,z,t) = \Phi(y,z) \sin \alpha_n x \exp(i\omega t), \]

(22)

where \( \alpha_n = \pi n/a \).

Substituting the seeking solutions in differential equations for \( \varphi \) potential, we will get

\[ \left( \frac{\partial^2 \Phi}{\partial y^2} - \alpha_n^2 \Phi \right) + a_1 \frac{\partial^2 \Phi}{\partial z^2} = -a_2 (w''_0 - \alpha_n^2 w_0). \]

(23)

Let’s seek (23) differential equation \( \Phi(y,z) = \varphi_0(y,z) + R(y) \). The seeking solution inserting (23), for \( \Phi(y,z) \) we will get

\[ \Phi(y,z) = (A_1 \exp(hy) + A_2 \exp(-hy)) (B_1 \sin(\beta_m z) + B_2 \cos(\beta_m z)) + C_1(y) \exp(\alpha_n y) + C_2(y) \exp(-\alpha_n y), \]

(24)

where \( C_1(y) = -\frac{a_2}{2\alpha_n} (w''_0 + \alpha_n w_0) + C_{10}; \quad C_2(y) = -\frac{a_2}{2\alpha_n} (w''_0 - \alpha_n w_0) + C_{20}; \quad \beta_m = \frac{2\pi m}{h}; \)

\( b = \alpha_n^2 + \lambda^2 a_1 \).

Satisfying boundary conditions on the border of \( z = \pm h \), for \( \varphi \) potential we will get

\[ \varphi(x,y,z) = (C_1(y) \exp(\alpha_n y) + C_2(y) \exp(-\alpha_n y)) (1 - \cos(\beta_m z) \sin \alpha_n x \exp(i\omega t)). \]

(25)

Substituting the seeking solution in the equation of bending vibration, we will get

\[ w''_0 - 2\alpha_n^2 \left( 1 + \frac{\gamma}{1 - \chi} \right) w''_0 + \alpha_n^4 \left( 1 - \frac{\Omega^2 - 2\gamma}{1 - \chi} \right) = 0, \]

(26)

where

\[ \gamma = \frac{3D_{13}a_2(1 + v)}{2h^2 \alpha_n^4}, \quad \chi = D_{15}a_2 \left( 1 + \frac{3}{h^2 \beta_m^2} \right), \quad \Omega^2 = \frac{2P h a_2^2}{\alpha_n^2 D}. \]

(27)

The solution of (26) differential equation will have following view
where $\theta_{1,2} = \pm \sqrt{1 - \zeta + \xi^2 + \hat{\Omega}^2}$, $\theta_{3,4} = \pm \sqrt{1 - \zeta - \xi^2 + \hat{\Omega}^2}$, $\zeta = \frac{\gamma}{1 - \chi}$, $\hat{\Omega}^2 = \frac{\Omega^2}{1 - \chi}$.

According to the condition, from (21) we will get
\[
 w_0(y) = B \exp(-\alpha_n y) + D \exp(-\alpha_p y),
\]
(29)

According to the damping condition, when $y \to \infty$, $w = 0$, $\varphi = 0$, the seeking constants $C_{10}$, $A$, $D$ are equal to zero. Then satisfying boundary conditions on $y = 0$ we will get system of mathematical equations towards the seeking unknown constants.

In order the system of algebraic equations has non-zero solution, it is necessary that its determinant was equal to zero. From the condition that the determinant will get system of mathematical equations towards the seeking unknown constants.

According to the damping condition from (30) follows that $\theta_1 > 0$, $\theta_3 > 0$, it follows from here $0 < \hat{\Omega}^2 < 1 - 2\zeta$. In limited cases the following inequalities must be satisfied. In the case when $\hat{\Omega}^2 = 0$, $K(0) > 0$, we will get $\theta_1 = 0$, $\theta_3 = \sqrt{1 - 2\zeta}$,
\[
 1 - 2\zeta + 2(1 - \nu)\sqrt{1 - 2\zeta} - \nu(\nu - 2\zeta) - r\left(\theta_1 \theta_3 + \theta_1 + \theta_3 + \nu\right) = 0.
\]
(31)

In the case when $\hat{\Omega}^2 = 0$, $K(1 - 2\zeta) < 0$, we will get $\theta_1 = \sqrt{2(1 - \zeta)}$, $\theta_3 = 0$.
\[
 -\nu(\nu - 2\zeta) - r\left(\sqrt{2(1 - \zeta)} + \nu\right) < 0.
\]
(32)

In the case, when piezoeffect is missing, (31) and (32) inequalities are satisfied, we will get known result [7]. For piezoceramic plate $2\ h$ with constant thickness and polarized along the normal of the middle plane of the plate, numerical analysis are done for defining the frequency of localized vibrations. The meaning of the frequency of localized vibrations dependant on dimensionless values $\gamma$ and $\chi$ at $\nu = 0.3$ are presented in the Table.

**Frequencies of localized vibrations**

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\chi$</th>
<th>$K(0)$</th>
<th>$K(1 - 2\zeta)$</th>
<th>$\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.15</td>
<td>1.62</td>
<td>-0.20</td>
<td>0.79</td>
</tr>
<tr>
<td>0.15</td>
<td>0.10</td>
<td>1.07</td>
<td>-0.39</td>
<td>0.72</td>
</tr>
<tr>
<td>0.20</td>
<td>0.25</td>
<td>0.69</td>
<td>-0.38</td>
<td>0.52</td>
</tr>
<tr>
<td>0.25</td>
<td>0.30</td>
<td>0.19</td>
<td>-0.56</td>
<td>0.55</td>
</tr>
</tbody>
</table>

In the case when piezoeffect is missing, the meaning of the frequency of localized vibrations is $\Omega = 0.99$. From the Table follows, that for piezoceramic plate $2\ h$ with constant thickness and polarized along the normal of the middle plane of the
Localized Bending Vibrations of Piezoceramic Transverse Plate

The frequency of localized vibrations and with the increase of non-dimensional values $y$, $\chi$, the size of frequency of localized vibrations decreases.

From the figure, it is seen that the frequency of localized vibrations decreases with the increasing of geometric parameters of the plate.

**Conclusion.** For a piezoceramic plate polarized along the normal of the middle plane of the plate, the equations of planar and bending vibrations are obtained. In the frequent case, bending vibrations are considered, for which a dispersion equation is obtained for determining the frequency of localized vibrations. It is shown that for bending vibrations of a piezoceramic plate polarized along the normal of the middle plane of the plate, the frequency of localized vibrations decreases at the increasing of dimensionless values $y$, $\chi$.

**Received 20.05.2017**

**References**