A UNIQUENESS THEOREM FOR A NONLINEAR SINGULAR INTEGRAL EQUATION ARISING IN $p$-ADIC STRING THEORY

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We study a singular nonlinear integral equation on the real line that appear in $p$-adic string theory. A uniqueness theorem for this equation in certain class of odd functions is proved. At the end of the paper we give examples, satisfying the conditions of the formulated theorem.

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**Introduction.** In this paper we study the following boundary value problem:

\begin{equation}
\varphi''(x) = (\mu(x) - 1)\varphi''(x) + \int_{\mathbb{R}} K(x-t)\varphi(t)dt, \quad x \in \mathbb{R},
\end{equation}

\begin{equation}
\varphi(\pm \infty) = \pm 1,
\end{equation}

with respect to unknown measurable and odd function $\varphi(x)$ defined on $\mathbb{R}$, where $m$ and $n$ are given odd numbers and

\begin{equation}
m > 2n.
\end{equation}

\(\mu\) and $K$ are even functions defined on $\mathbb{R}$ and satisfy the following conditions

a) $\mu(0) = +\infty$, $\mu(x) \geq 1$, $x \in \mathbb{R}$, $\lim_{x \to \infty} \mu(x) = 1$;

b) $\mu - 1 \in \bigcap_{\mu=1}^{3} L_{\mu}(0, +\infty)$;

c) $K(x) \geq 0$, $x \in \mathbb{R}$, $K(x) \downarrow$ in $x$ on $\mathbb{R}^+ := [0, +\infty)$;

d) $K \in L_{1}(\mathbb{R}) \cap C_{M}(\mathbb{R})$, $\int_{-\infty}^{\infty} K(x)dx = 1$, $\int_{0}^{\infty} xK(x)dx < +\infty$, where $C_{M}(\mathbb{R})$ is the space of continuous and substantially bounded on $\mathbb{R}$ functions.

The Eq. (1) arises in $p$-adic closed-open string theory [1-6]. In particular, the boundary value problem (1), (2) describes rolling of tachyon’s open-closed $p$-adic strings. Recently in [7] it was proved that the boundary value problem (1), (2) under the conditions (3), (a)–(d), has a nontrivial odd solution on the real line besides

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we have
\[
\varphi(x) = \begin{cases} f(x), & \text{if } x > 0, \\ -f(-x), & \text{if } x < 0, \end{cases}
\]
where \( f(x) \) is the nonnegative nontrivial solution of the following nonlinear integral equation with a sum-difference kernel
\[
f^m(x) = (\mu(x) - 1)f^m(x) + \int_0^\infty (K(x-t) - K(x+t)) f(t)dt, \quad x > 0, \quad \lim_{x \to \infty} f(x) = 1. \tag{5}
\]
Moreover, the solution \( \varphi(x) \) has the following properties:
\begin{itemize}
  \item[i)] \( \psi(x) \leq \varphi(x) \leq (1 + M \frac{1}{\pi(t)} \mu^\frac{1}{2}(x)), \quad x > 0, \) where \( \psi(x) \) is the solution of the following boundary value problem
  \[
  \psi^m(x) = \int_0^\infty (K(x-t) - K(x+t)) \psi(t)dt, \quad x \in \mathbb{R}^+, \tag{6}
  \]
  \[
  \lim_{x \to \infty} \psi(x) = 1. \tag{7}
  \]
\end{itemize}
It is also established that \( \psi(x) \) is nonnegative, monotone increasing, continuous and bounded function. Moreover, \( \psi(0) = 0, \ 1 - \psi \in L_1(0, +\infty); \)
\begin{itemize}
  \item[ii)] \( 1 - \varphi \in L_1(0, +\infty), \ 1 + \varphi \in L_1(-\infty, 0); \)
  \item[iii)] \( \varphi(\pm 0) = \pm \infty. \)
\end{itemize}
Finally, \( M := \int_0^\infty (\mu(t) - 1)dt \cdot \sup_{x \in \mathbb{R}} K(x) < +\infty. \)

The main goal of this paper is to prove the uniqueness of the solution of boundary value problem \( \{1\}, \{2\} \) in certain class of odd functions.

**Uniqueness Theorem.** Below we will prove that boundary value problem \( \{1\}, \{2\} \) in the following class of odd measurable functions on \( \mathbb{R} \)
\[
\mathcal{M} := \left\{ \varphi : 1 \pm \varphi \in L_1(\mathbb{R}^+), \ 0 \leq \varphi(x) \leq (1 + M \frac{1}{\pi(t)} \mu^\frac{1}{2}(x)), \ x > 0 \right\}
\]
has unique solution.

From the result of work \( \{7\} \) it follows that in the above mentioned class of functions the boundary value problem \( \{1\}, \{2\} \) is equivalent to the boundary value problem \( \{6\}, \{7\} \). Hence it is enough to prove the uniqueness of the solution of boundary value problem \( \{6\}, \{7\} \) in the following class of nonnegative measurable functions on \( (0, +\infty) \).
\[
\mathcal{P} := \left\{ f : 1 - f \in L_1(\mathbb{R}^+), \ 0 \leq f(x) \leq (1 + M \frac{1}{\pi(t)} \mu^\frac{1}{2}(x)), \ x > 0 \right\}.
\]
We suppose to the contrary the given Eq. \( \{1\} \) in class \( \mathcal{P} \) has two different solutions \( f \) and \( \tilde{f} \). Then from the simple inequality
\[
0 \leq | f(x) - \tilde{f}(x) | \leq | 1 - f(x) | + | 1 - \tilde{f}(x) |
\]
and due to \( f, \tilde{f} \in \mathcal{P} \), it can be easily verified that \( f - \tilde{f} \in L_1(0, +\infty) \).

Notice that since \( f \) and \( \tilde{f} \) are nonnegative solutions of Eq. \( \{5\} \), and due to condition e) the following inequality holds:
\[
K(x-t) \geq K(x+t), \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+.
\]
Indeed, if $x \geq t \geq 0$, then by the monotonicity of $K$ it follows that $K(x-t) \geq K(x+t)$. If $0 \leq x \leq t$, then once again, using the monotonicity and evenness of $K$, we obtain

$$K(x-t) = K(t-x) \geq K(x+t).$$

Taking into account the properties of function $\psi$ (see above), from inequality (4) and (5) we get

$$f^{m-n}(x) > \mu(x) - 1, \quad \tilde{f}^{m-n}(x) > \mu(x) - 1, \quad x > 0. \quad (8)$$

We estimate the difference

$$|f^m(x) - \tilde{f}^m(x)| \leq (\mu(x) - 1) |f^n(x) - \tilde{f}^n(x)| \int_0^\infty (K(x-t) - K(x+t)) (f(t) - \tilde{f}(t)) \, dt$$

or

$$|f(x) - \tilde{f}(x)| \left\{ f^{m-1}(x) + f^{m-2}(x) \tilde{f}(x) + \cdots + f(x) \tilde{f}^{m-2}(x) + \tilde{f}^{m-1}(x) \right\} \leq$$

$$\leq (\mu(x) - 1) |f(x) - \tilde{f}(x)| \left\{ (f^{m-1}(x) + f^{m-2}(x) \tilde{f}(x) + \cdots +

+ f(x) \tilde{f}^{m-2}(x) + \tilde{f}^{m-1}(x) \right\} + \int_0^\infty (K(x-t) - K(x+t)) (f(t) - \tilde{f}(t)) \, dt. \quad (9)$$

We will prove that the right hand side of the obtained inequality (9) belongs to the space $L_1(0, +\infty)$. For this purpose we first multiply both sides of (9) by $f(x)$ and separately prove that

$$J_1 := (\mu(x) - 1) |f(x) - \tilde{f}(x)| \left\{ f^n(x) + f^{n-1}(x) \tilde{f}(x) + \cdots +

+ f^2(x) \tilde{f}^{n-2}(x) + f(x) \tilde{f}^{n-1}(x) \right\} \in L_1(\mathbb{R}^+), \quad (10)$$

$$J_2 := f(x) \int_0^\infty (K(x-t) - K(x+t)) (f(t) - \tilde{f}(t)) \, dt \in L_1(\mathbb{R}^+). \quad (11)$$

First of all we will prove inclusion (10). Due to the relation $f, \tilde{f} \in \mathcal{P}$, taking into account (4), property (i) and triangle inequalities, we get

$$0 \leq |J_1| \leq 2(\mu(x) - 1)(1 + M)^{\frac{n+1}{n}} \mu^{\frac{1}{n}}(x) \left\{ f^n(x) + f^{n-1}(x) \tilde{f}(x) + \cdots +

+ f^2(x) \tilde{f}^{n-2}(x) + f(x) \tilde{f}^{n-1}(x) \right\} \leq 2n(1 + M)^{\frac{n+1}{n}} (\mu(x) - 1) \mu^{\frac{n+1}{n}}(x) \leq$$

$$\leq 2n(1 + M)^{\frac{n+1}{n}} \left\{ (\mu(x) - 1)^3 + 2(\mu(x) - 1)^2 + (\mu(x) - 1) \right\} \in L_1(\mathbb{R}^+),$$

which immediately implies $J_1 \in L_1(\mathbb{R}^+)$. Now we prove that $J_2 \in L_1(\mathbb{R}^+)$. Using conditions b)-d) and (i), we have

$$0 \leq |J_2| \leq (1 + M)^{\frac{1}{n}} (\mu^{\frac{1}{n}}(x) - 1) \int_0^\infty (K(x-t) - K(x+t)) (f(t) - \tilde{f}(t)) \, dt +

+ (1 + M)^{\frac{1}{n}} \int_0^\infty (K(x-t) - K(x+t)) (f(t) - \tilde{f}(t)) \, dt \leq$$
\[
\leq 2(1 + M) \frac{2}{\alpha - \gamma} (\mu(x) - 1) \int_0^\infty (K(x-t) - K(x+t)) \mu(t) dt +
\]
\[
+ (1 + M) \frac{1}{\alpha - \gamma} \int_0^\infty (K(x-t) - K(x+t)) |f(t) - \tilde{f}(t)| dt \leq
\]
\[
\leq 2(1 + M) \frac{2}{\alpha - \gamma} (\mu(x) - 1) \int_0^\infty (K(x-t) - K(x+t)) (\mu(t) - 1) dt +
\]
\[
+ 2(1 + M) \frac{2}{\alpha - \gamma} (\mu(x) - 1) \int_0^\infty (K(x-t) - K(x+t)) dt +
\]
\[
+ (1 + M) \frac{1}{\alpha - \gamma} \int_0^\infty (K(x-t) - K(x+t)) |f(t) - \tilde{f}(t)| dt \leq
\]
\[
\leq 2(1 + M) \frac{2}{\alpha - \gamma} (\mu(x) - 1) +
\]
\[
+ (1 + M) \frac{1}{\alpha - \gamma} \int_0^\infty (K(x-t) - K(x+t)) |f(t) - \tilde{f}(t)| dt.
\]

Observe that first two terms in last expression belong to space \( L_1(\mathbb{R}^+) \). Since \( f - \tilde{f} \in L_1(\mathbb{R}^+) \) and \( K \in L_1(\mathbb{R}) \cap C_M(\mathbb{R}) \), according to Fubini’s theorem [8], we get
\[
\int_0^\infty (K(x-t) - K(x+t)) |f(t) - \tilde{f}(t)| dt \in L_1(\mathbb{R}^+).
\]

Hence \( J_2 \in L_1(\mathbb{R}^+) \). Since \( J_1 \in L_1(\mathbb{R}^+) \), \( J_2 \in L_1(\mathbb{R}^+) \), from (9) it follows that
\[
|f(x) - \tilde{f}(x)| (f^m(x) + f^{m-1}(x)\tilde{f}(x) + \cdots + f^2(x)\tilde{f}^{m-2}(x) + f(x)\tilde{f}^{m-1}(x)) \in L_1(\mathbb{R}^+).
\]

After multiplying both sides of inequality (9) by function \( f(x) \) we can integrate obtained inequality on \((0, +\infty)\). So we get
\[
\int_0^\infty |f(x) - \tilde{f}(x)| (f^m(x) + f^{m-1}(x)\tilde{f}(x) + \cdots + f^2(x)\tilde{f}^{m-2}(x) + f(x)\tilde{f}^{m-1}(x)) dx \leq
\]
\[
\leq \int_0^\infty |f(x) - \tilde{f}(x)| (\mu(x) - 1)(f^n(x) + f^{n-1}(x)\tilde{f}(x) + \cdots +
\]
\[
+ f^2(x)\tilde{f}^{n-2}(x) + f(x)\tilde{f}^{n-1}(x)) dx +
\]
\[
+ \int_0^\infty f(x) \int_0^\infty (K(x-t) - K(x+t)) |f(t) - \tilde{f}(t)| dt dx.
\]

Since the kernel \( K \) is an even function, taking into account (1) and Fubini’s theorem, we obtain
\[
\int_0^\infty f(x) \int_0^\infty (K(x-t) - K(x+t)) |f(t) - \tilde{f}(t)| dt dx =
\]
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\[ \begin{align*}
&= \int_0^\infty |f(t) - \tilde{f}(t)| \int_0^\infty (K(x-t) - K(x+t))f(x)dt \ dx = \\
&= \int_0^\infty |f(t) - \tilde{f}(t)| \int_0^\infty (K(t-x) - K(x+t))f(x)dt \ dx = \\
&= \int_0^\infty |f(t) - \tilde{f}(t)| (f^n(t) - (\mu(t) - 1)f^n(t))dt.
\end{align*} \]

Thus, considering last the relation in (12), we get
\[ \int_0^\infty |f(x) - \tilde{f}(x)| (f^{m-1}(x)\tilde{f}(x) + \ldots +
+f^2(x)\tilde{m}-2(x) + f(x)\tilde{m}-1(x) - (\mu(x) - 1)f^{m-1}(x)\tilde{f}(x) - \ldots -
-(\mu(x) - 1)f^2(x)\tilde{m}-2(x) - (\mu(x) - 1)f(x)\tilde{m}-1(x))dx \leq 0. \]

This in turn implies
\[ \int_0^\infty |f(x) - \tilde{f}(x)| \left\{ f^{m-1}(x)\tilde{f}(x)(f^{m-n}(x) - (\mu(x) - 1)) + \ldots + f^2(x)\tilde{m}-2(x)\right. \times
\left. (f^{m-n}(x) - (\mu(x) - 1)) + f(x)\tilde{m}-1(x)(f^{m-n}(x) - (\mu(x) - 1)) \right\}dx \leq 0. \]  

From (13) and (8) it follows that \( f(x) = \tilde{f}(x) \) almost everywhere on \((0, +\infty)\), since the function
\[ f^{m-1}(x)\tilde{f}(x)(f^{m-n}(x) - (\mu(x) - 1)) + \ldots +
+f^2(x)\tilde{m}-2(x)(f^{m-n}(x) - (\mu(x) - 1)) + f(x)\tilde{m}-1(x)(f^{m-n}(x) - (\mu(x) - 1)) \]
is positive on \((0, +\infty)\).

Thus the following theorem holds.

**Theorem.** Let the conditions \((3), \text{a)}-\text{d)}\) be satisfied. Then the boundary value problem \((1), \text{2)}\) in class \(M\) of measurable functions has a unique odd solution.

**Examples.** At the end of the work we present several examples of functions \(K\) and \(\mu\), for which all conditions of the formulated theorem hold:

1. \( K(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}; \)
2. \( K(x) = \frac{\alpha}{2} e^{-\alpha|x|}, \alpha > 0; \)
3. \( K(x) = \int_a^b e^{-|s|}G(s)ds, \) where \( a > 0, b \leq +\infty, G(s) > 0, s \in [a, b], \)
   \[ \int_a^b \frac{G(s)ds}{s} = \frac{1}{2}, G(s) \in L_1(a, b); \]
4. \( \mu(x) = 1 + \frac{e^{-x^2}}{|x|^{\alpha}}, \alpha \in \left(0, \frac{1}{3}\right), \ x \in \mathbb{R}; \)

5. \( \mu(x) = 1 + \frac{e^{-|x|}}{|x|^{\frac{1}{2}}}, \ x \in \mathbb{R}; \)

6. \( \mu(x) = 1 + \frac{1}{|x|^{\alpha}} \cdot \frac{1}{1 + x^4}, \ x \in \mathbb{R}. \)

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