ON THE UNIQUENESS OF $\beta\delta$-NORMAL FORM OF TYPED $\lambda$-TERMS
FOR THE CANONICAL NOTION OF $\delta$-REDUCTION

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In this paper we consider a substitution and inheritance property, which is
the necessary and sufficient condition for the uniqueness of $\beta\delta$-normal form
of typed $\lambda$-terms, for canonical notion of $\delta$-reduction. Typed $\lambda$-terms use
variables of any order and constants of order $\leq 1$, where the constants of order
1 are strongly computable, monotonic functions with indeterminate values of
arguments. The canonical notion of $\delta$-reduction is the notion of $\delta$-reduction
that is used in the implementation of functional programming languages.


Keywords: canonical notion of $\delta$-reduction, SI-property, $\beta\delta$-normal form.

Introduction. The definitions of this section can be found in [1-3]. Let $M$
be a partially ordered set, which has a least element $\perp$, which corresponds to the
indeterminate value, and each element of $M$ is comparable only with $\perp$ and with
itself. Let us define the set of types (denoted by Types) in the conventional way
(see [2]). Let $\alpha \in$ Types and $V_{\alpha}$ be a countable set of variables of type $\alpha$, then
$V = \bigcup_{\alpha \in$ Types} $V_{\alpha}$ is the set of all variables. The set of all terms is denoted by
$\Lambda = \bigcup_{\alpha \in$ Types} $\Lambda_{\alpha}$, where $\Lambda_{\alpha}$ is the set of terms of type $\alpha$ and is defined in the con-
ventional way (see [2]). The notions of free and bound occurrences of variables in
terms as well as the notion of a free variable are introduced in the conventional way
too [2]. The set of all free variables in the term $t$ is denoted by $FV(t)$. Terms $t_1$ and
$t_2$ are said to be congruent (which is denoted by $t_1 \equiv t_2$), if one term can be obtained
from the other by renaming bound variables.

Further, we assume that $M$ is a recursive set and the considered terms use
variables of any order and constants of order $\leq 1$, where the constants of order 1
are strongly computable, monotonic functions with indeterminate values of
arguments [1].

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A term of the form \( \lambda x_1 \ldots x_k [\tau](t_1, \ldots, t_k) \), where \( x_i \in V_\alpha, i \neq j \Rightarrow x_i \neq x_j \), \( \tau \in \Lambda, t_i \in \Lambda_\alpha, \alpha \in \text{Types}, i, j = 1, \ldots, k, k \geq 1 \), is called a \( \beta \)-redex, its convolution is the term \( \tau\{x_1/x_1, \ldots, x_k/x_k\} \), where \( \{x_1/x_1, \ldots, x_k/x_k\} \) is a substitution (the substitution as well as the admissible application of substitution are defined in a conventional way \([2]\)). The set of all pairs \((\tau_0, \tau_1)\), where \( \tau_0 \) is a \( \beta \)-redex and \( \tau_1 \) is its convolution, is called a notion of \( \beta \)-reduction. A one-step \( \beta \)-reduction \((\rightarrow) \) and \( \beta \)-reduction \((\rightarrow\beta) \) are defined in the conventional way. A term containing no \( \beta \)-redexes is called a \( \beta \)-normal form. The set of all \( \beta \)-normal forms is denoted by \( \beta-NF \).

A \( \delta \)-redex has a form \( f(t_1, \ldots, t_k) \), where \( f \in [M^k \to M], t_i \in \Lambda M, i = 1, \ldots, k \), \( k \geq 1 \), its convolution is either \( m \in M \) and in this case \( f(t_1, \ldots, t_k) \sim m \) or a subterm \( t_i \) and in this case \( f(t_1, \ldots, t_k) \sim t_i, i = 1, \ldots, k \). A fixed set of term pairs \((\tau_0, \tau_1)\), where \( \tau_0 \) is a \( \delta \)-redex and \( \tau_1 \) is its convolution, is called a notion of \( \delta \)-reduction \((\rightarrow\delta) \) as well as \( \rightarrow_{\beta\delta} \) and \( \rightarrow_{\beta\delta} \) are defined in the conventional way \([2]\)). A term containing no \( \beta \delta \)-redexes is called normal form. The set of all normal forms is denoted by \( NF \).

**Definition 1.** \([2]\) An effective, single-valued notion of \( \delta \)-reduction is called a canonical notion of \( \delta \)-reduction, if

1. \( t \in \beta-NF, t \sim m, m \in M \setminus \{\bot\} \Rightarrow t \rightarrow_{\delta} m \),
2. \( t \in \beta-NF, FV(t) = \emptyset, t \sim \bot \Rightarrow t \rightarrow_{\delta} \bot \).

**Definition 2.** The notion of \( \delta \)-reduction has the substitution property (S-property), if from \((f(t_1, \ldots, t_k), \tau) \in \delta \), where \( t_1, \ldots, t_k, \tau \in \Lambda M, f \in [M^k \to M], FV(f(t_1, \ldots, t_k)) \neq \emptyset, k \geq 1 \), and from the following properties

S1. \( f(t_1, \ldots, t_k) \) is not constant term and \( \tau \equiv t_j, 1 \leq j \leq k \), or
S2. \( f(t_1, \ldots, t_k) \sim \bot \) and \( \tau \equiv t_j, 1 \leq j \leq k \), or
S3. \( f(t_1, \ldots, t_k) \sim \bot \) and \( \tau \equiv \bot \),

it follows that for each admissible application of substitution \( \{x_1/x_1, \ldots, x_n/x_n\} \) (shortly \( \{\tau/x\} \)), where \( x_i \in V_\alpha, x_i \in V_\alpha, x_i \in \text{Types}, i \neq j \Rightarrow x_i \neq x_j \), \( i, j = 1, \ldots, n, n \geq 0 \), there exist terms \( t'_1, \ldots, t'_k \) such that \( t_1 \{\tau/x\} \rightarrow t'_1, \ldots, t_k \{\tau/x\} \rightarrow t'_k \) and \((f(t'_1, \ldots, t'_k), t'_j) \in \delta \) if \( \tau \equiv t_j \) and \((f(t'_1, \ldots, t'_k), \bot) \in \delta \) if \( \tau \equiv \bot \).

**Definition 3.** The notion of \( \delta \)-reduction has the inheritance property (I-property), if from \((f(t_1, \ldots, t_k), \tau) \in \delta \), where \( t_1, \ldots, t_k, \tau \in \Lambda M, f \in [M^k \to M], FV(f(t_1, \ldots, t_k)) \neq \emptyset, k \geq 1 \) and \( t_i \equiv \mu_r \) for some \( i (1 \leq i \leq k) \), where \( r \) is a redex and from the following properties:

I1. \( f(t_1, \ldots, t_k) \) is not constant term and \( \tau \equiv t_j, 1 \leq j \leq k \), or
I2. \( f(t_1, \ldots, t_k) \sim \bot \) and \( \tau \equiv t_j, 1 \leq j \leq k \), or
I3. \( f(t_1, \ldots, t_k) \sim \bot \) and \( \tau \equiv \bot \),

it follows that there exist terms \( t'_1, \ldots, t'_k \in \Lambda M \) such that \( t_1 \rightarrow t'_1, \ldots, \mu_r \rightarrow t'_1, \ldots, t_k \rightarrow t'_k \) and \((f(t'_1, \ldots, t'_k), t'_j) \in \delta \) if \( \tau \equiv t_j \) and \((f(t'_1, \ldots, t'_k), \bot) \in \delta \) if \( \tau \equiv \bot \), where \( r' \) is the convolution of the redex \( r \).

**Definition 4.** The canonical notion of \( \delta \)-reduction has SI-property, if it has S-property and I-property.

**Two Canonical Notions of \( \delta \)-Reduction.** Let \( M = N \cup \{\bot\} \), where \( N = \{0, 1, 2, \ldots\} \) and \( C = \{\text{add, min, max, inc, dec}\} \), \( C^c = C \cup \{\not\text{eq, numbers}\} \), where \( \text{inc, dec} \in [M \to M], \text{add, min, max, not_eq, numbers} \in [M^2 \to M] \) and for every
\( m, m_1, m_2 \in M \) we have:

\[
add(m_1, m_2) = \begin{cases} 
m_1 + m_2, & \text{if } m_1, m_2 \in N, \\
\bot, & \text{otherwise.}
\end{cases}
\]

\[
min(m_1, m_2) = \begin{cases} 
m_1, & \text{if } m_1, m_2 \in N \text{ and } m_1 \leq m_2, \\
m_2, & \text{if } m_1, m_2 \in N \text{ and } m_1 > m_2, \\
\bot, & \text{otherwise.}
\end{cases}
\]

\[
max(m_1, m_2) = \begin{cases} 
m_2, & \text{if } m_1, m_2 \in N \text{ and } m_1 \leq m_2, \\
m_1, & \text{if } m_1, m_2 \in N \text{ and } m_1 > m_2, \\
\bot, & \text{otherwise.}
\end{cases}
\]

\[
inc(m) = \begin{cases} 
m + 1, & \text{if } m \in N, \\
\bot, & \text{if } m = \bot.
\end{cases}
\]

\[
dec(m) = \begin{cases} 
0, & \text{if } m \in N \text{ and } m = 0, \\
m - 1, & \text{if } m \in N \text{ and } m \geq 1, \\
\bot, & \text{if } m = \bot.
\end{cases}
\]

\[
not_eq(m_1, m_2) = \begin{cases} 
1, & \text{if } m_1, m_2 \in N \text{ and } m_1 \neq m_2, \\
\bot, & \text{otherwise.}
\end{cases}
\]

\[
numbers(m_1, m_2) = \begin{cases} 
1, & \text{if } m_1, m_2 \in N, \\
\bot, & \text{otherwise.}
\end{cases}
\]

It is easy to see that all functions of the set \( C' \) are strong computable, naturally extended functions with indeterminate values of arguments (a function is said to be naturally extended, if its value is \( \bot \) whenever the value of at least one of the arguments is \( \bot \)). Let us consider the notion of \( \delta \)-reduction \( \delta \) for the set \( C' \):

- \( (add(n_1, n_2), n) \in \delta \), where \( n_1, n_2, n \in N \) and \( n = n_1 + n_2 \)
- \( (add(\bot, t), \bot) \in \delta \), where \( t \in \Lambda \)
- \( (add(t, \bot), \bot) \in \delta \), where \( t \in \Lambda \)
- \( (min(n_1, n_2), n_1) \in \delta \), where \( n_1, n_2 \in N \) and \( n_1 \leq n_2 \)
- \( (min(n_1, n_2), n_2) \in \delta \), where \( n_1, n_2 \in N \) and \( n_1 > n_2 \)
- \( (min(\bot, t), \bot) \in \delta \), where \( t \in \Lambda \)
- \( (min(t, \bot), \bot) \in \delta \), where \( t \in \Lambda \)
- \( (max(n_1, n_2), n_2) \in \delta \), where \( n_1, n_2 \in N \) and \( n_1 \leq n_2 \)
- \( (max(n_1, n_2), n_1) \in \delta \), where \( n_1, n_2 \in N \) and \( n_1 > n_2 \)
- \( (max(\bot, t), \bot) \in \delta \), where \( t \in \Lambda \)
- \( (max(t, \bot), \bot) \in \delta \), where \( t \in \Lambda \)
- \( (inc(n_1), n_2) \in \delta \), where \( n_1, n_2 \in N \) and \( n_2 = n_1 + 1 \)
- \( (inc(\bot), \bot) \in \delta \)
- \( (dec(n_1), n_2) \in \delta \), where \( n_1, n_2 \in N, n_1 > 0 \) and \( n_2 = n_1 - 1 \)
Let us consider $\delta'$ notion of $\delta$-reduction for the set $C'$:

$$(t, \tau) \in \delta \Rightarrow (t, \tau) \in \delta'$, where $t, \tau \in \Lambda M$$

(not_eq(n_1, n_2), 1) \in \delta', where $n_1, n_2 \in \mathbb{N}$ and $n_1 \neq n_2$

(not_eq(t, t), 1) \in \delta', where $t \in \Lambda M$

(not_eq(t, \bot), 1) \in \delta', where $t \in \Lambda M$

(numbers(n_1, n_2), 1) \in \delta', where $n_1, n_2 \in \mathbb{N}$

(numbers(\bot, t), 1) \in \delta', where $t \in \Lambda M$

(numbers(\bot, \bot), 1) \in \delta', where $t \in \Lambda M$.

It is easy to see that $\delta$ and $\delta'$ are canonical notions of $\delta$-reduction.

**S-Property.** We say that a notion of $\delta$-reduction does not hold only point $S_i$, $i = 1, 2, 3$, if it holds all points of $S$-property and $I$-property, except point $S_i$ of $S$-property.

Let $\delta_1 = \delta \cup \{(\max(\text{inc}(x), x), \text{inc}(x)) \mid x \in V_M\}$. It is easy to see that $\delta_1$ is an effective, single valued notion of $\delta$-reduction. Since $\delta \subset \delta_1$, $\delta_1$ is a canonical notion of $\delta$-reduction.

**Proposition 1.** For the canonical notion of $\delta$-reduction $\delta_1$ the following properties hold:

a) $\delta_1$ does not hold only the point $S_1$;

b) there exists a term that has two different normal forms.

**Proof.**

a) To show that $\delta_1$ has $I$-property, let us consider all pairs $(f(t_1, \ldots, t_k), \tau) \in \delta_1$ such that $f(t_1, \ldots, t_k)$ is non constant term or $f(t_1, \ldots, t_k) \sim \bot$, where $f \in C$, $\text{FV}(f(t_1, \ldots, t_k)) \neq \emptyset$, $k = 1, 2$, $t_i \equiv \mu_r$ for some $i (1 \leq i \leq k)$, where $r$ is a redex, and $\tau \equiv t_j$ for some $j (1 \leq j \leq k)$, or $\tau \equiv \bot$. The following cases are possible:

i) $f \in \{\text{add}, \text{min}, \text{max}\}$, $t_1 \equiv \bot, t_2 \equiv \mu_r \in \Lambda M$, where $r$ is a redex, $k = 2, i = 2$ and $\tau \equiv \bot$. Since $(f(\bot, t), \bot) \in \delta_1$ for every $t \in \Lambda M$, we have $(f(\bot, \mu_r), \bot) \in \delta_1$, where $r'$ is the convolution of the redex $r$.

ii) $f \in \{\text{add}, \text{min}, \text{max}\}$, $t_1 \equiv \mu_r \in \Lambda M$, where $r$ is a redex, $t_2 \equiv \bot, k = 2, i = 1$ and $\tau \equiv \bot$. Since $(f(t, \bot), \bot) \in \delta_1$ for every $t \in \Lambda M$, we have $(f(\mu_r, \bot), \bot) \in \delta_1$, where $r'$ is the convolution of the redex $r$.

Therefore $\delta_1$ has $I$-property. To show that $\delta_1$ does not hold only the point $S_1$, let us consider all pairs $(f(t_1, \ldots, t_k), \tau) \in \delta_1$ such that $f(t_1, \ldots, t_k)$ is non constant term or $f(t_1, \ldots, t_k) \sim \bot$, where $f \in C$, $\text{FV}(f(t_1, \ldots, t_k)) \neq \emptyset$, $k = 1, 2$, and $\tau \equiv t_j$ for some $j (1 \leq j \leq k)$ or $\tau \equiv \bot$. The following cases are possible:

i) $f \in \{\text{add}, \text{min}, \text{max}\}$, $t_1 \equiv \bot, t_2 \equiv \Lambda M$, $k = 2$ and $\tau \equiv \bot$, where $\text{FV}(t_2) \neq \emptyset$.

Since $(f(\bot, t), \bot) \in \delta_1$ for any $t \in \Lambda M$, then $(f(\bot, t_2 \sigma), \bot) \in \delta_1$ for any admissible application of the substitution $\sigma$.

ii) $f \in \{\text{add}, \text{min}, \text{max}\}$, $t_1 \equiv \Lambda M, t_2 \equiv \bot, k = 2$ and $\tau \equiv \bot$, where $\text{FV}(t_1) \neq \emptyset$.

Since $(f(t, \bot), \bot) \in \delta_1$ for any $t \in \Lambda M$, we get $(f(t_1 \sigma, \bot), \bot) \in \delta_1$ for any admissible application of the substitution $\sigma$. 
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iii) $f \equiv \text{max}$, $t_1 \equiv \text{inc}(x)$, $t_2 \equiv x$, $k = 2$ and $\tau \equiv t_1$, where $x \subseteq V_M$. For the admissible application of the substitution $\sigma = \{\text{inc}(y)/x\}$ we have: $t_1 \sigma \equiv t_2 \sigma \equiv \text{inc}(x) \{\text{inc}(y)/x\} \equiv \text{inc}(\text{inc}(y)) \in \mathcal{NF}$, $t_2 \sigma \equiv X \{\text{inc}(y)/x\} \equiv \text{inc}(y) \in \mathcal{NF}$. Since $f(t_1, t_2)$ is a non constant term and $(f(t_1 \sigma, t_2 \sigma), t \sigma) \notin \delta_1$, $\delta_1$ does not hold the point S1.

Since the S-property violated only in the case (iii), where $f(t_1, t_2)$ is the non constant term, $\delta_1$ does not hold only the point S1.

b) Let us show that for $\delta_1$ the term $\lambda x[\text{max}(\text{inc}(x), x)][\text{inc}(y)]$ has two different normal forms.

\[
\lambda x[\text{max}(\text{inc}(x), x)][\text{inc}(y)] \rightarrow_{\delta_1} \lambda x[\text{inc}(x)][\text{inc}(y)] \rightarrow_{\beta} \text{inc}(\text{inc}(y)) \in \mathcal{NF};
\]

\[
\lambda x[\text{max}(\text{inc}(x), x)][\text{inc}(y)] \rightarrow_{\beta} \text{max}(\text{inc}(y)), \text{inc}(y) \in \mathcal{NF}.
\]

Let $\delta_2 = \delta' \cup \{(\text{numbers}(\text{not} \_ \text{eq})(\text{dec}(\text{inc}(x)), x), 0), \text{not} \_ \text{eq}(\text{dec}(\text{inc}(x)), x)\}$ $\in \mathcal{V}_M$. It is easy to see that $\delta_2$ is an effective, single valued notion of $\delta$-reduction. Since $\delta' \subset \delta_2$, then $\delta_2$ is a canonical notion of $\delta$-reduction.

**Proposition 2.** For the canonical notion of $\delta$-reduction $\delta_2$ the following hold true:

a) $\delta_2$ does not hold only the point S2;

b) there exists a term that has two different normal forms.

**Proof.**

a) To show that $\delta_2$ has I-property, let us consider all pairs $(f(t_1, \ldots, t_k), \tau) \in \delta_2$ such that $f(t_1, \ldots, t_k)$ is non constant term or $f(t_1, \ldots, t_k) \sim \bot$, where $f \in C'$, $\text{FV}(f(t_1, \ldots, t_k)) \neq \emptyset$, $k = 1, 2$, $t_i \equiv t_i$, $i = 1, \ldots, k$, where $r$ is a redex, and $\tau \equiv t_j$, $1 \leq j \leq k$ or $\tau \equiv \bot$. The following cases are possible:

i) $f \in \{\text{add}, \text{min}, \text{max}, \text{not} \_ \text{eq}, \text{numbers}\}$, $t_1 \equiv \bot$, $t_2 \equiv \mu_r$, $t_i \equiv \mu_r$, for some $i (1 \leq i \leq k)$, where $r$ is a redex, and $\tau \equiv t_j$, $1 \leq j \leq k$ or $\tau \equiv \bot$. The following cases are possible:

i) $f \in \{\text{add}, \text{min}, \text{max}, \text{not} \_ \text{eq}, \text{numbers}\}$, $t_1 \equiv \bot$, $t_2 \equiv \mu_r$, $t_i \equiv \mu_r$, for some $i (1 \leq i \leq k)$, where $r$ is a redex, and $\tau \equiv t_j$, $1 \leq j \leq k$ or $\tau \equiv \bot$. The following cases are possible:

Therefore $\delta_2$ has I-property. To show that $\delta_2$ does not hold only the point S2, let us consider all pairs $(f(t_1, \ldots, t_k), \tau) \in \delta_2$ such that $f(t_1, \ldots, t_k)$ is the non constant term or $f(t_1, \ldots, t_k) \sim \bot$, where $f \in C'$, $\text{FV}(f(t_1, \ldots, t_k)) \neq \emptyset$, $k = 1, 2$. The following cases are possible:

i) $f \in \{\text{add}, \text{min}, \text{max}, \text{not} \_ \text{eq}, \text{numbers}\}$, $t_1 \equiv \bot$, $t_2 \equiv \Lambda$, $k = 2$ and $\tau \equiv \bot$, where $f \in C'$, $\text{FV}(f(t_1, \ldots, t_k)) \neq \emptyset$, $k = 1, 2$. The following cases are possible:

ii) $f \in \{\text{add}, \text{min}, \text{max}, \text{not} \_ \text{eq}, \text{numbers}\}$, $t_1 \equiv t_2 \equiv \mu_r$, $t_i \equiv \mu_r$, for some $i (1 \leq i \leq k)$, where $r$ is a redex, and $\tau \equiv t_j$, $1 \leq j \leq k$ or $\tau \equiv \bot$. The following cases are possible:
iii) \( f \equiv \text{not}_e q, t_1 \equiv t_2 \in \Lambda_M, k = 2 \) and \( \tau \equiv \bot \). Since \( (f(t), \bot) \in \delta_t \) for every \( t \in \Lambda_M \), \( (f(t_1 \sigma, t_2 \sigma), \bot) \in \delta_t \) for any admissible application of the substitution \( \sigma \).

iv) \( f \equiv \text{numbers}, t_1 \equiv \text{not}_e q(\text{dec}(\text{inc}(x)), x) \), where \( x \in V_M, t_2 \equiv 0, k = 2 \) and \( \tau \equiv t_1 \). For the admissible application of substitution \( \sigma = \{ \text{add}(x, 2)/x \} \) we have:
\[
t_1 \sigma \equiv \text{not}_e q(\text{dec}(\text{inc}(\text{add}(x, 2))), \text{add}(x, 2)) \equiv t'_1 \in \text{NF} \quad \text{and} \quad t_2 \sigma \equiv 0 \equiv t'_2 \in \text{NF}.
\]
Since \( f(t_1, t_2) \sim \bot \) and \( \text{numbers}(t'_1, t'_2, t'_3) \notin \delta_2 \), \( \delta_2 \) does not hold the point S2.

Therefore \( \delta_3 \) does not hold only the point S2.

b) Let us show that for \( \delta_t \) the term
\[
t' \equiv \lambda_{xy}[\text{numbers}(\text{not}_e q(\text{dec}(\text{inc}(x)), x), 0)](\text{add}(x, 2))
\]
has two different normal forms:
\[
\begin{align*}
t' \rightarrow_{\beta} & \text{numbers}(\text{not}_e q(\text{dec}(\text{inc}(\text{add}(x, 2))), \text{add}(x, 2)), 0) \in \text{NF}; \\
t' \rightarrow_{\delta_t} & \lambda_{xy}[\text{not}_e q(\text{dec}(\text{inc}(x)), x)](\text{add}(x, 2)) \rightarrow_{\beta} \\
& \text{not}_e q(\text{dec}(\text{inc}(\text{add}(x, 2))), \text{add}(x, 2)) \in \text{NF}. \quad \Box
\end{align*}
\]

Let \( \delta_3 = \delta' \cup \{ (\text{numbers}(\text{not}_e q(\text{dec}(x), x), 0), \bot) \mid x \in V_M \} \). It is easy to see that \( \delta_3 \) is an effective, single valued notion of \( \delta \)-reduction. Since \( \delta' \subset \delta_3 \), then \( \delta_3 \) is a canonical notion of \( \delta \)-reduction.

**Proposition 3.** For the canonical notion of \( \delta \)-reduction \( \delta_3 \) the following holds:

a) \( \delta_3 \) does not hold only the point S3;

b) there exists a term that has two different normal forms.

**Proof.**

a) It can be shown that \( \delta_3 \) has I-property as shown in Proposition 2. To show that \( \delta_3 \) holds the S1, S2 points and does not hold the point S3, let us consider all pairs \( (f(t_1, ..., t_k), \tau) \in \delta_3 \) such that \( f(t_1, ..., t_k) \) is a non constant term or \( f(t_1, ..., t_k) \sim \bot \), where \( f \in C', t_1, ..., t_k, \tau \in \Lambda_M, f \in [M^k \rightarrow M], FV(f(t_1, ..., t_k)) \neq \emptyset, k = 1, 2 \). The following cases are possible:

i) \( f \in \{ \text{add}, \text{min}, \text{max}, \text{not}_e q, \text{numbers} \}, t_1 \equiv \bot, t_2 \in \Lambda_M, k = 2 \) and \( \tau \equiv \bot \);

ii) \( f \in \{ \text{add}, \text{min}, \text{max}, \text{not}_e q, \text{numbers} \}, t_1 \in \Lambda_M, t_2 \equiv \bot, k = 2 \) and \( \tau \equiv \bot \);

iii) \( f \equiv \text{not}_e q, t_1 \equiv t_2 \in \Lambda_M, k = 2 \) and \( \tau \equiv \bot \).

We can show that S-property is true in (i) – (iii) cases, as shown in Proposition 2.

iv) \( f \equiv \text{numbers}, t_1 \equiv \text{not}_e q(\text{dec}(\text{inc}(x)), x), x \), where \( x \in V_M, t_2 \equiv 0, k = 2 \) and \( \tau \equiv \bot \). For the admissible application of the substitution \( \sigma = \{ \text{add}(x, 2)/x \} \) we have:
\[
t_1 \sigma \equiv \text{not}_e q(\text{dec}(\text{inc}(\text{add}(x, 2))), \text{add}(x, 2)) \equiv t'_1 \in \text{NF} \quad \text{and} \quad t_2 \sigma \equiv 0 \equiv t'_2 \in \text{NF}.
\]
Since \( f(t_1, t_2) \sim \bot \) and \( (f(t'_1, t'_2, \bot) \notin \delta_3, \delta_3 \) does not hold the point S3.

Therefore \( \delta_3 \) does not hold only the point S3.

b) Let us show that for \( \delta_t \) the term
\[
t \equiv \lambda_{xy}[\text{numbers}(\text{not}_e q(\text{dec}(\text{inc}(x)), x), 0)](\text{add}(x, 2))
\]
has two different normal forms:
\[
\begin{align*}
t \rightarrow_{\delta_t} & \lambda_{xy}[\bot](\text{add}(x, 2)) \rightarrow_{\beta} \bot \in \text{NF}; \\
t \rightarrow_{\delta_t} & \text{numbers}(\text{not}_e q(\text{dec}(\text{inc}(\text{add}(x, 2))), \text{add}(x, 2)), 0) \in \text{NF}. \quad \Box
\end{align*}
\]

**I-Property.** We say that a notion of \( \delta \)-reduction does not hold only the point II, \( i = 1, 2, 3 \), if it holds all points of S-property and I-property, except the point II of I-property. If \( t \in \beta - \text{NF} \), \( t \sim m, m \in M \), then \( t \equiv m \) or \( t \equiv f(t_1, ..., t_k) \), where
$f \in [M^k \to M], t_i \in \Lambda_M, t_i \in \beta - NF, i = 1, \ldots, k, k \geq 1$. We introduce the notion of rank for such terms: $\text{rank}(m) = 0, \text{rank}(f(t_1, \ldots, t_k)) = 1 + \max(\text{rank}(t_1), \ldots, \text{rank}(t_k))$.

Let $\delta_4 = \delta \cup \{(\min(\text{add}(\tau_1, \tau_2), \text{add}(\tau_2, \tau_1)), \text{add}(\tau_1, \tau_2)) \mid \tau_1, \tau_2 \in \Lambda_M) \cup \{(\min(\text{add}(\tau_1, \tau_2), \min(\text{add}(\tau_2, \tau_1), \text{add}(\tau_1, \tau_2))), \min(\text{add}(\tau_2, \tau_1), \text{add}(\tau_1, \tau_2))) \mid \tau_1, \tau_2 \in \Lambda_M\}$. It is easy to see that $\delta_4$ is an effective, single valued notion of $\delta$-reduction. Since $\delta \subset \delta_4$, then $\delta_4$ is a canonical notion of $\delta$-reduction.

**Proposition 4.** For the canonical notion of $\delta$-reduction $\delta_4$ the following takes place:

a) $\delta_4$ does not hold only the point P1;

b) there exists a term that has two different normal forms. To prove Proposition 4 let us prove Lemma 1.

**Lemma 1.** For the canonical notion of $\delta$-reduction $\delta$ and for any term $t \in \Lambda$, we have: if $t \sim \perp$, then $t \rightarrow_\beta \perp$.

**Proof.** Let $t \in \Lambda$, $t \sim \perp$ and $t \rightarrow_\beta t' \in \beta - NF$. Therefore, $t' \sim \perp$. If $\text{rank}(t') = 0$, then $t' \equiv \perp$, $t' \rightarrow_\beta \perp$ and $t \rightarrow_\beta \perp$. If $\text{rank}(t') = 1$, then the following cases are possible:

i) $t' \equiv f(m)$, where $f \in \{\text{inc}, \text{dec}\}, m \in M$. Since $f(m) \sim \perp$, we have $m \equiv \perp$, $(t', \perp) \in \delta, t' \rightarrow_\delta \perp$ and $t \rightarrow_\beta \perp$;

ii) $t' \equiv f(m_1, m_2)$, where $f \in \{\text{add}, \text{min}, \text{max}\}, m_1, m_2 \in M$. Since $f(m_1, m_2) \sim \perp$, we have $m_1 \equiv \perp$ or $m_2 \equiv \perp$. Therefore, $(f(m_1, m_2), \perp) \in \delta, t' \rightarrow_\delta \perp, t \rightarrow_\beta \perp$.

Let $\text{rank}(t') = n > 1$, then $t' \equiv f(t_1, \ldots, t_k)$, where $f \in C$ and $t_i \in \Lambda_M$, $t_i \in \beta - NF$, $i = 1, \ldots, k, k \geq 1$, and we suppose that $\tau, \sigma \in \beta - NF$ and $\tau \sim \perp$, then $\tau \rightarrow_\beta \perp$, where $\delta \subset \delta_4$, and for any $t \in \Lambda$. Following cases are possible:

i) $t' \equiv f(\tau)$, where $f \in \{\text{inc}, \text{dec}\}, \tau \in \Lambda_M, \tau \in \beta - NF$. Since $f(\tau) \sim \perp$, then $\tau \sim \perp$. Since $\text{rank}(\tau) = n - 1$, by the induction hypothesis, $\tau \rightarrow_\beta \perp$. Therefore, $f(\tau) \rightarrow_\beta f(\perp) \rightarrow_\delta \perp$ and $t \rightarrow_\beta \perp$;

ii) $t' \equiv f(t_1, t_2)$, where $f \in \{\text{add}, \text{min}, \text{max}\}, t_1, t_2 \in \Lambda_M, t_1, t_2 \in \beta - NF$. Since $f(t_1, t_2) \sim \perp$, we can say $t_1 \sim \perp$ or $t_2 \sim \perp$. Without loss of generality we suppose that $t_1 \sim \perp$. Since $\text{rank}(t_1) < n$, by the induction hypothesis, $t_1 \rightarrow_\beta \perp$. Therefore, $f(t_1, t_2) \rightarrow_\beta f(\perp, t_2) \rightarrow_\delta \perp$ and $t \rightarrow_\beta \perp$. \hfill $\square$

**Proof of Proposition 4.**

a) If $(t, \tau) \in \delta_4$, then $(t \sigma, \tau \sigma) \in \delta_4$ for every admissible application of substitution $\sigma$. Therefore $\delta_4$ has S-property. Let us show that $\delta_4$ does not hold the point P1. Let $f \equiv \text{min}, t_1 \equiv \text{add}(x,y), t_2 \equiv \text{min}(\text{add}(y,x), \text{add}(x,y))$, then $(f(t_1, t_2), t_2) \in \delta_4$. It is easy to see that $f(t_1, t_2)$ is a non constant term. Since $t_1 \equiv \text{add}(x,y) \equiv t'_1 \in NF$, $t_2 \rightarrow \delta_4 \text{add}(y,x) \equiv t'_2 \in NF$ and $(f(t'_1, t'_2), t'_2) \notin \delta_4$, then $\delta_4$ does not hold the point P1.

Let $(f(t_1, \ldots, t_k), \tau) \in \delta_4$, where $t_1, \ldots, t_k, \tau \in \Lambda_M, f \in C, \text{FV}(f(t_1, \ldots, t_k)) \neq \emptyset$, $k = 1, 2, t_i \equiv \mu_r$ for some $i (1 \leq i \leq k)$, where $r$ is a redex and $f(t_1, \ldots, t_k) \sim \perp$. Following cases are possible:

i) $f \in \{\text{add}, \text{min}, \text{max}\}, t_1 \equiv \perp, t_2 \equiv \mu_r \in \Lambda_M, r$ is a redex, $k = 2, i = 2, \tau \equiv \perp$;
ii) $f \in \{\text{add}, \text{min}, \text{max}\}, t_1 \equiv \mu_r \in \Lambda_M, r$ is a redex, $t_2 \equiv \bot$, $k = 2$, $i = 1$, $\tau \equiv \bot$. It can be shown that $I$-property is true in the (i), (ii) cases, as shown in Proposition 1.

iii) $f = \text{min}, t_1 \equiv \text{add}(\tau_1, \tau_2) t_2 \equiv \text{add}(\tau_2, \tau_1), \tau \equiv t_1, i = 1, 2$.

iv) $f = \text{min}, t_1 \equiv \text{add}(\tau_1, \tau_2) t_2 \equiv \text{min}(\text{add}(\tau_2, \tau_2), \text{add}(\tau_1, \tau_2)), \tau \equiv t_2, i = 1, 2$.

Since in both (iii), (iv) cases $f(t_1, t_2) \sim \bot$, so $\tau \sim \bot$. It is easy to see that $t_1 \sim t_2 \sim \bot$. Without lose of generality we suppose that $i = 1$. Therefore, $t_1 \equiv \mu_r \rightarrow \beta_\delta$ $\mu_r \sim \bot$, where $r'$ is the convolution of the redex $r$. Since $\delta \subset \delta_1$, from Lemma 1 follows that $\mu_r \rightarrow \beta_\delta \bot$, $t_2 \rightarrow \beta_\delta \bot$ and $\tau \rightarrow \beta_\delta \bot$. Since $(f(\bot, \bot), \bot) \in \delta_i$, $\delta_i$ holds the points 12 and 13. Therefore $\delta_i$ does not hold only the point 11.

b) Let us show that for $\delta_1$ the term $t' \equiv \min(\text{add}(x, y), \text{min}(\text{add}(y, x), \text{add}(x, y)))$ has two different normal forms:

t' \rightarrow \delta_1 \min(\text{add}(x, y), \text{add}(x, y)) \rightarrow \delta_1 \text{add}(x, y) \in \text{NF};
t' \rightarrow \delta_2 \min(\text{add}(x, y), \text{add}(x, y)) \rightarrow \delta_2 \text{add}(x, y) \in \text{NF}.

Let $\delta_3 = \delta' \cup \{\text{not eq}(\text{add}(\tau_1, \tau_2), \text{add}(\tau_1, \tau_2), \text{add}(\tau_2, \tau_1)), \text{not eq}(\text{add}(\tau_1, \tau_2), \text{add}(\tau_2, \tau_1))\}$.

It is easy to see that $\delta_3$ is an effective, single valued notion of $\delta$-reduction. Since $\delta' \subset \delta_3$, $\delta_3$ is a canonical notion of $\delta$-reduction.

**Proposition 5.** For the canonical notion of $\delta$-reduction $\delta_3$ the following properties hold:

a) $\delta_3$ does not hold only the point 12;

b) there exists a term that has two different normal forms.

**Proof.**

a) If $(t, \tau) \in \delta_3$, then $(t\sigma, \tau\sigma) \in \delta_3$ for every admissible application of substitution $\sigma$. Therefore $\delta_3$ has $S$-property.

To show that $\delta_3$ does not hold only the point 12, let us consider all pairs $(f(t_1, \ldots, t_k), \tau) \in \delta_3$ such that $f(t_1, \ldots, t_k)$ is non constant term or $f(t_1, \ldots, t_k) \sim \bot$, where $f \in C', \text{FV}(f(t_1, \ldots, t_k)) \neq \emptyset$, $k = 1, 2$, $t_i \equiv \mu_r$ for some $i$ ($1 \leq i \leq k$), where $r$ is a redex, and $\tau \equiv t_j, 1 \leq j \leq k$ or $\tau \equiv \bot$. The following cases are possible:

i) $f \in \{\text{add}, \text{min}, \text{max}, \text{not eq}, \text{numbers}\}, t_1 \equiv \bot, t_2 \equiv \mu_r \in \Lambda_M$, where $r$ is a redex, $k = 2$ and $\tau \equiv \bot$;

ii) $f \in \{\text{add}, \text{min}, \text{max}, \text{not eq}, \text{numbers}\}, t_1 \equiv \mu_r \in \Lambda_M$, where $r$ is a redex, $t_2 \equiv \bot$, $k = 2$ and $\tau \equiv \bot$;

iii) $f \equiv \text{not eq}, t_1 \equiv t_2 \equiv \mu_r \in \Lambda_M$, where $r$ is a redex, $k = 2$ and $\tau \equiv \bot$.

We can show that I-property is true in (i), (ii), (iii) cases, as shown in Proposition 2.

iv) $f \equiv \text{min}, t_1 \equiv \text{add}(\tau_1, \tau_2), t_2 \equiv \text{add}(\tau_2, \tau_1), \tau \equiv t_1$, where $\tau_1, \tau_2 \in \Lambda_M$.

If $f(t_1, t_2) \sim \bot$, then $t_1 \sim t_2 \sim \bot$. Without loss of generality suppose that $i = 1$.

If $t_1 \equiv r$ and $r'$ is its convolution, then from the definition of $\delta_3$ it follows that $\rho \equiv \bot$, $t_2$ is a redex and $r'$ is its convolution. Therefore, $t_1 \rightarrow_{\delta_3} r' \equiv \bot, t_2 \rightarrow_{\delta_3} r' \equiv \bot$ and $(f(\bot, \bot), \bot) \in \delta_3$. If $t_1 \neq r$, then $\tau_1 \equiv \tau_1, t_2 \equiv \text{add}(\tau_2, \tau_2), t_2 \equiv \text{add}(\tau_2, \tau_2), \text{add}(\tau_2, \tau_2) \in \delta_3$, or $t_2 \equiv t_2, (\text{min}(\text{add}(\tau_2, \tau_2), \text{add}(\tau_2, \tau_2)))$.

Let $f(t_1, t_2)$ be a non constant term. Without loss of generality, suppose that $t_1$ is a
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redex. Then $t_1 \rightarrow_{\delta} t'_1$ and from the definition of $\delta$ it follows that $t'_1 \equiv m \in M$, $t_2 \rightarrow_{\delta} t'_2 \equiv m$. Therefore, $f(t_1, t_2) \rightarrow_{\beta\delta} f(m, m) \rightarrow_{\delta} m$ and $f(t_1, t_2) \sim m$, which is a contradiction. Therefore, $t_1$ and $t_2$ cannot be redexes. Without loss of generality suppose $t_1 \equiv t_{1r}$, where $r$ is a redex. It is easy to see that $(\min(\text{add}(t_{1r}, t_2), \text{add}(t_2, t_{1r})), \text{add}(t_{1r}, t_2)) \in \delta$, where $r'$ is the convolution of redex $r$.

\[ v) \quad f \equiv \text{not}_{eq} \quad t_1 \equiv \text{not}_{eq}(\min(\text{add}(t_1, t_2), \text{add}(t_2, t_1)), \text{add}(t_2, t_1)) \]

\[ t_2 \equiv \text{not}_{eq}(\text{add}(t_1, t_2), \text{add}(t_2, t_1)), \tau \equiv t_1, \text{ where } t_1, t_2 \in \Lambda_M. \]

It is easy to see that $f(t_1, t_2) \sim \bot$. Let $i = 1$, $t_1 \equiv x$, $t_2 \equiv y$, $r \equiv \min(\text{add}(x, y), \text{add}(y, x))$, where $x, y \in V_M$. Since $t_1 \equiv \tau \equiv \mu, \mu_r \equiv \text{not}_{eq}(\text{add}(x, y), \text{add}(y, x)) \in NF$ and $(f(\mu_r, t_2), \mu_r) \notin \delta$, $\delta$ does not hold the point $12$.

Therefore $\delta$ does not hold only the point $12$.

b) Let us show that for the $\delta$ term $t_5 \equiv \text{not}_{eq}(\text{not}_{eq}(\min(\text{add}(t_1, t_2), \text{add}(t_2, t_1)), \text{add}(t_2, t_1)))$ has two different normal forms: $t_5 \rightarrow_{\delta} \text{not}_{eq}(\min(\text{add}(t_1, t_2), \text{add}(t_2, t_1)), \text{add}(t_2, t_1)) \rightarrow_{\delta} \text{not}_{eq}(\min(\text{add}(t_1, t_2), \text{add}(t_2, t_1))) \in NF$.

\[ t_5 \rightarrow_{\delta} \text{not}_{eq}(\text{not}_{eq}(\text{add}(t_1, t_2), \text{add}(t_2, t_1)), \text{not}_{eq}(\text{add}(t_1, t_2), \text{add}(t_2, t_1))) \rightarrow_{\delta} \bot \in NF; \]

Let $\delta = \delta' \cup \{ \min(\text{add}(t_1, t_2), \text{add}(t_2, t_1)), \text{add}(t_2, t_1)) \mid t_1, t_2 \in \Lambda_M \} \cup \{ \text{add}(t_1, t_2), \text{add}(t_2, t_1)) \mid t_1, t_2 \in \Lambda_M \}$. It is easy to see that $\delta$ is an effective, single valued notion of $\delta$-reduction. Since $\delta' \subset \delta$, then $\delta$ is a canonical notion of $\delta$-reduction.

**Proposition 6.** For the canonical notion of $\delta$-reduction $\delta$ the following hold:

a) $\delta$ does not hold only the I3 point;

b) there exists a term that has two different normal forms.

**Proof.**

a) If $(t, \tau) \in \delta$, then $(t\sigma, \tau\sigma) \in \delta$ for every admissible application of the substitution $\sigma$. Therefore $\delta$ has S-property.

To show that $\delta$ does not hold only the point I3, let us consider all pairs $(f(t_1, \ldots, t_k), \tau) \in \delta$ such that $f(t_1, \ldots, t_k)$ is non constant term or $f(t_1, \ldots, t_k) \sim \bot$, where $f \in \mathcal{C}$, $\text{FV}(f(t_1, \ldots, t_k)) \neq \emptyset$, $k = 1, 2, t_i \equiv \mu_r$ for some $i$ $(1 \leq i \leq k)$, where $r$ is a redex, and $\tau \equiv \tau_j, 1 \leq j \leq k$ or $\tau \equiv \bot$. The following cases are possible:

i) $f \in \{ \text{add}, \text{min}, \text{max}, \text{not}_{eq}, \text{numbers} \}, t_1 \equiv \bot, t_2 \equiv \mu_r \in \Lambda_M$, where $r$ is a redex, $k = 2$ and $\tau \equiv \bot$;

ii) $f \in \{ \text{add}, \text{min}, \text{max}, \text{not}_{eq}, \text{numbers} \}, t_1 \equiv \mu_r \in \Lambda_M$, where $r$ is a redex, $t_2 \equiv \bot, k = 2$ and $\tau \equiv \bot$;

iii) $f \equiv \text{not}_{eq}$, $t_1 \equiv t_2 \equiv \mu_r \in \Lambda_M$, $k = 2$ and $\tau \equiv \bot$, where $r$ is a redex;

iv) $f \equiv \text{min}$, $t_1 \equiv \text{add}(t_1, t_2), t_2 \equiv \text{add}(t_2, t_1), \tau \equiv \bot$, where $t_1, t_2 \in \Lambda_M$.

It can be shown that I-property is true in (i)-(iv) cases, as shown in Proposition 5.

v) $f \equiv \text{not}_{eq}$, $t_1 \equiv \text{min}(\text{add}(x, y), \text{add}(y, x)), t_2 \equiv \text{add}(x, y), k = 2, i = 1$ and $\tau \equiv \bot$. Since $t_1 \rightarrow_{\delta} \text{add}(x, x) \in NF, \text{add}(y, y) \in NF$ and $(f(\text{add}(x, y), \text{add}(x, y)), \bot) \notin \delta$, $\delta$ does not hold the point I3. Therefore $\delta$ does not hold only the point I3.
b) Let us show that for $\delta_6$ the term $\text{not}_{eq}(\text{min}(\text{add}(x,y),\text{add}(y,x)),\text{add}(x,y))$ \\
$\sim \bot$ has two different normal forms:

$\text{not}_{eq}(\text{min}(\text{add}(x,y),\text{add}(y,x)),\text{add}(x,y)) \rightarrow_{\delta_6} \text{not}_{eq}(\text{add}(y,x),\text{add}(x,y)) \in \text{NF};$

$\text{not}_{eq}(\text{min}(\text{add}(x,y),\text{add}(y,x)),\text{add}(x,y)) \rightarrow_{\delta_6} \bot \in \text{NF}.$

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