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Mathematics

ON EULER TYPE EQUATION

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In the present paper an Euler type equation is considered and it is proved that for $\alpha \in [0,1)$ $(\alpha \in (2n-1, 2n])$ the characteristic polynomial has $2n$ real roots.

For other values of α the issue concerning the number of the real roots of this polynomial is investigated.

***Keywords*:** Euler type equation, Hardy's inequality, oscillation problems, characteristic polynomial.

We consider an Euler type equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} - \gamma_{n,\alpha} t^{\alpha-2n} y = 0\,, \tag{1}$$

where $t \geq 0$, $\alpha \geq 0$, $\alpha \neq 1,3,...,2n-1$ and $\gamma_{n,\alpha} = 4^{-n} \prod_{i=1}^{n} (2i-1-\alpha)^2$.

Note that the numbers $\gamma_{n,\alpha}$ appear naturally, when one uses Hardy's inequality to estimate the scalar product $(t^\alpha y^{(n)}, y^{(n)})$ (see [1]). In [2] the following statement was proved: the spectrum of the operator generated by the differential expression $t^{2n-\alpha} L : L_{2,\alpha-2n} \to L_{2,\alpha-2n}$, where $Ly \equiv (-1)^n (t^\alpha y^{(n)})^{(n)}$ and $L_{2,\alpha-2n} = \left\{ f, \int_0^b t^{\alpha-2n} |f(t)|^2 \, dt < \infty \right\}$, is purely continuous and coincides with the ray $\sigma(t^{2n-\alpha} L) = [\gamma_{n,\alpha}, +\infty)$. The real roots of the characteristic polynomial $p(\lambda) = (-1)^n \prod_{i=0}^{n-1} (\lambda - i)(\lambda - n + \alpha - i) - \gamma_{n,\alpha}$. Of the equation (1) were used in [3] to study the oscillation problems concerning the equation

$$Ly \equiv (-1)^n (t^\alpha y^{(n)})^{(n)} - \gamma_{n,\alpha} t^{\alpha-2n} y = q(t) y\,.$$

Note that the numbers $0,1,...,n-1$ and $2n-1-\alpha, 2n-2-\alpha,...,n-\alpha$ are symmetrical with respect to $\dfrac{2n-1-\alpha}{2}$. After changing the variable

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$\mu = \dfrac{2n-1-\alpha}{2} - \lambda$ the polynomial $p(\lambda)$ gets the form

$$\tilde{p}(\mu) = (-1)^n \prod_{i=1}^{n} \left(\frac{2i-1-\alpha}{2} - \mu \right)\left(\frac{-2i+1+\alpha}{2} - \mu \right) - \gamma_{n,\alpha} =$$

$$= \frac{(-1)^n}{4^n} \prod_{l=1}^{n} (4\mu^2 - (2i-1-\alpha)^2) - \gamma_{n,\alpha}.$$

After changing the variable $x = 4\mu^2$ from the equation $\tilde{p}(\mu) = 0$ we get

$$f(x) = \prod_{i=1}^{n} (x - (2i-1-\alpha)^2) - (-1)^n (1-\alpha)^2 (3-\alpha)^2 \cdots (2n-1-\alpha)^2 = 0 . \quad (2)$$

Note that in general case the equation (2) has no n real roots. For example, when $n = 3$ the equation (2) has the form

$$x(x^2 - ((1-\alpha)^2 + (3-\alpha)^2 + (5-\alpha)^2)x + (1-\alpha)^2(3-\alpha)^2 +$$
$$+ (3-\alpha)^2(5-\alpha)^2 + (1-\alpha)^2(5-\alpha)^2) = 0.$$

It is easy to calculate the discriminant of the quadratic equation:

$$D = -3\alpha^4 + 36\alpha^3 - 114\alpha^2 + 36\alpha + 189 = -3(a+1)(a-7)(a-3)^2,$$

and, therefore, the quadratic equation for any α with large absolute value has two complex roots.

Note also that $x = 0$ for any value of α is a root of (2), and (2) doesn't change its form when one replaces α with $2n - \alpha$. Therefore, it is enough to consider only the case $\alpha \le n$.

Let us now prove that for $\alpha \in [0,1)$ the equation (2) alongside with the root $x = 0$ has also ($n-1$) positive roots. First we prove this statement for even numbers, i.e. for $n = 2k$. Let

$$g(x) = (x - (1-\alpha)^2)(x - (3-\alpha)^2) \cdots (x - (2n-1-\alpha)^2) .$$

Obviously,

$$f((1-\alpha)^2) = f((3-\alpha)^2) = ... = f((4k-1-\alpha)^2) = -(1-\alpha)^2 \cdots (4k-1-\alpha)^2 < 0 . \quad (3)$$

Now by induction we prove that $f((4i)^2) > 0,\ i = 1,2,...,k.$ For $i = 1$ we have

$$f(4^2) = \prod_{i=2}^{2k-3} (2i+1-\alpha)^2 (7-\alpha)^2 (1-\alpha)(3-\alpha)(4k-3-\alpha)(4k-1-\alpha) \times$$
$$\times 32k(3 - \alpha^2 + 4\alpha k) \quad (4)$$

and, therefore, $f(4^2) > 0$. Now assuming $f((4i)^2) > 0$, we prove that $f((4(i+1))^2) > 0$. It is enough to prove that $f((4i+4)^2) > f((4i)^2)$, which is equivalent to the inequality $\dfrac{g((4i+4)^2)}{g((4i)^2)} > 1$. Further

$$\frac{g((4i+4)^2)}{g((4i)^2)} = \frac{((4i+4)^2 - (1-\alpha)^2)((4i+4)^2 - (3-\alpha)^2) \cdots ((4i+4)^2 - (4k-1-\alpha)^2)}{((4i)^2 - (1-\alpha)^2)((4i)^2 - (3-\alpha)^2) \cdots ((4i)^2 - (4k-1-\alpha)^2)} =$$

$$= \frac{(4i+3+\alpha)(4i+1+\alpha)}{(4i+3-\alpha)(4i+1-\alpha)} \cdot \frac{(4i+4k+1-\alpha)(4k+4i+3-\alpha)}{(4i-4k+1+\alpha)(4i-4k+3+\alpha)} > 1,$$

since $(4i+3-\alpha)(4i+1-\alpha)>0$ and $(4i-4k+1+\alpha)(4i-4k+3+\alpha)>0$. Thus we obtain that $f((4i)^2)>0, i=1,2,...,k$. Now using inequality (3) and the first Theorem of Bolzano–Cauchy, we conclude that the equation (2) has $2k-1$ positive roots alongside with the root $x=0$. Using the form of the polynomial $\tilde{p}(\mu)$ and changing the variable $\mu=\frac{2n-1-\alpha}{2}-\lambda$, we conclude that the polynomial $p(\lambda)$ has $2n$ real roots, whereas $\frac{2n-1-\alpha}{2}$ is a double root and the other roots are symmetrical with respect to $\frac{2n-1-\alpha}{2}$. For odd numbers $n=2k+1$ the proof is similar.

Now consider the case $1<\alpha\le n$. We'll consider the cases $\alpha\in(4j-1,4j+1)$ and $\alpha\in(4j-3,4j-1)$ separately. Let $n=2k$ and $\alpha\in(4j-3,4j-1)$. We'll show that $f((4i+2)^2)>0, \quad i=0,1,...,k-1$.

For $i=0$ we have

$$f(2^2)=(3-\alpha)^2(5-\alpha)^2\cdots(4k-3-\alpha)^2(\alpha-1)(4k-1-\alpha)((1+\alpha)(4k+1-\alpha)-\\-(\alpha-1)(4k-1-\alpha)),$$

that implies $f(2^2)>0$. Now we obtain as we did above

$$\frac{g((4i+6)^2)}{g((4i+2)^2)}=\frac{(4i+5+\alpha)(4i+3+\alpha)}{(4i+5-\alpha)(4i+3-\alpha)}\cdot\frac{(4k+4i+3-\alpha)(4k+4i+5-\alpha)}{(4k-4i-3-\alpha)(4k-4i-5-\alpha)}>1,$$

since for $\alpha\in(4j-3,4j-1)$ the denominators of the both fractions are positive and obviously, both fractions are greater than one.

Note that $(4k-1-\alpha)^2\in((4k-4j)^2, (4k-4j+2)^2)$ and taking into consideration the inequalities (3) and $f((4i+2)^2)>0$, we conclude that (2) has at least $n-2j+1$ positive roots alongside with the root $x=0$. Hence the polynomial $p(\lambda)$ has at least $2n-4j+4$ real roots. Note that for $j=1$, i.e. for $\alpha\in(1,3)$, the situation is similar to the case $\alpha\in[0,1)$.

For $n=2k$ and $\alpha\in(4j-1, 4j+1)$ instead of the points $x=(4i+2)^2$ we take $x=(4i)^2, i=1,2,...,k$. Then we prove that $f((4i)^2)>0$ and conclude that the equation (2) has at least $n-2j-1$ real roots alongside with the root $x=0$. Thus the polynomial $p(\lambda)$ has at least $2n-4j$ real roots.

Now we consider the case of odd numbers $n=2k+1$ and $\alpha\in(4j-1, 4j+1)$.

As in the case $\alpha\in[0,1)$, we prove that $f(4^2)<0$. Note that for $i<k-j$ the inequality

$$\frac{g((4i+4)^2)}{g((4i)^2)}=\frac{(4i+3+\alpha)(4i+1+\alpha)}{(4i+3-\alpha)(4i+1-\alpha)}\cdot\frac{(4i+4k+3-\alpha)(4i+4k+5-\alpha)}{(4i-(4k-1)+\alpha)(4i-(4k+1)+\alpha)}>1$$

immediately follows from the positiveness of both denominators. For $i=k-j$ the denominator of the second fraction $(4i-(4k-1)+\alpha)(4i-(4k+1)+\alpha)$ is negative $\big((4i+3-\alpha)(4i+1-\alpha)>0$ for $\alpha\in(4j-1,4j+1)$ and any $i\big)$, i.e. by induction in i we prove that $f((4i)^2)<0,\ i=1,2,...,k-j.$ As a result we conclude that the polynomial $p(\lambda)$ for $\alpha\in(4j-1,4j+1)$ has at least $2n-4j-2$ real roots. We prove analogously that the polynomial $p(\lambda)$ for $\alpha\in(4j-3,4j-1)$ has at least $2n-4j+2$ real roots.

Unfortunately, similar considerations for $\alpha>2n$ do not lead to the desired result. For instance, for $n=2k$ we get $f(4^2)<0$, since in the expression (4) $3-\alpha^2+4\alpha k<0.$

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Էյլերի տիպի մի հավասարման մասին

Դիտարկվում է Էյլերի տիպի մի հավասարման բնութագրիչ բազմանդամը և ցույց է տրվում, որ $\alpha \in [0,1) \left(\alpha \in (2n-1, 2n]\right)$ դեպքում այն ունի $2n$ իրական արմատներ, իսկ մյուս α-ների համար հետազոտվում է իրական արմատների քանակի հարցը:

С.А. Осипова, Л.П. Тепоян

Об одном уравнении типа Эйлера

В работе рассматривается уравнение типа Эйлера и доказывается, что при $\alpha \in [0,1) \left(\alpha \in (2n-1, 2n]\right)$ характеристический полином имеет $2n$ действительных корней, а для остальных значений α исследуется вопрос о количестве этих корней.