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Mathematics

ON CONNECTION OF ONE CLASS OF ONE-DIMENSIONAL PSEUDODIFFERENTIAL OPERATORS WITH SINGULAR INTEGRAL OPERATORS

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The paper discusses a homogeneous one-dimensional pseudodifferential equation with a symbol of the form

$$A(x,\xi)=A_0(\xi)+\sum_{k=1}^{N}\operatorname{th}\frac{\pi}{\alpha}\left(x-\lambda_k+i\frac{\alpha\beta}{2}\right)A_k(\xi)$$

($x,\xi,\lambda_k\in\mathbb{R}$, $\alpha>0$, $-1<\beta<1$, $k=1,2,\ldots,N$),

where $A_k(\xi)$ $(k=0,1,\ldots,N)$ are locally integrable functions from class of symbols of non-negative order r .

The method of bringing the pseudodifferential equation to a system of one-dimensional singular integral equations with Cauchy's kernel is proposed.

***Keywords*:** pseudodifferential operator, factorization of matrix-function.

1^0. The paper is devoted to the study of one-dimensional pseudodifferential operators $A(x,D)=F^{-1}_{\xi\to x}A(x,\xi)F_{x\to\xi}$ with $A(x,\xi)=1+A_0(\xi)+\sum\limits_{k=1}^{N}\varphi(x,\lambda_k)A_k(\xi)$ $(x,\xi\in\mathbb{R})$ symbols, where $\varphi(x,\lambda_k)=\operatorname{th}\frac{\pi}{\alpha}\left(x-\lambda_k+i\frac{\alpha\beta}{2}\right)$ $(x\in\mathbb{R},\ \alpha>0,-1<\beta<1,$ $\lambda_k\in\mathbb{R},\ \ k=1,2,...,N)$. It is assumed that $A_k(\xi)\in S_r^0$ $(k=0,1\ldots,N)$ for some $r\geq 0$, where S_r^0 is a set of locally integrable on $\mathbb{R}$ functions f, which correspond to the following condition: $|f|\leq c\left(1+|\xi|\right)^r$.

Earlier, the cases of $r=0$ and $N=1$ had been discussed in [1] and [2] correspondingly. In this paper the study of solvability of equation $A(x,D)y=0$ is brought to solvability of some characteristic system of one-dimensional singular integral equations with Cauchy's kernel. Theory of matrix singular integral operators is currently quite well developed (e.g. see [3–7]).

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2^0. Let $H_r(\mathbb{R})$ $(r \in \mathbb{R})$ be the Sobolev–Slobodetsky space of generalized functions u, the Fourier transformation $\hat{u}$ of which belongs to the space $L_2(\mathbb{R},(1+|x|)^r)$. Below, we denote by Λ_u the operator of multiplication by matrix-function u $(\Lambda_u y = uy)$. It is well known (e.g. see [8]) that a pseudodifferential operator $A(x,D)$ with symbol $A(x,\xi)$: $A(x,D)u = \int\limits_{-\infty}^{\infty} e^{-ix\xi} A(x,\xi)\hat{u}(\xi)d\xi$ $\left(u \in C_0^\infty(\mathbb{R})\right)$ may be extended up to a continuous operator from $H_r(\mathbb{R})$ to $L_2(\mathbb{R})$. We assume below that $A(x,D)$ acts on $H_r(\mathbb{R})$. For a function y, defined on $\mathbb{R}_+ = (0,+\infty)$ through $\gamma_k y$ $(k = 0,1,...,N)$, we introduce the function $e^{s_k x} y(e^{\alpha x})$, where $\sigma = \alpha(\beta+1)/2$ and $s_k = \sigma + i\lambda_k$ $(\lambda_0 \in \mathbb{R})$. It is obvious, that the operators γ_k are continuous mappings from $L_2(\mathbb{R}_+,\rho)$ to $L_2(\mathbb{R})$, where $\rho(t) = t^\beta (-1 < \beta < 1)$ for $t \ge 0$.

$h(x) = \alpha^{-1} \ln x$ $(x \in \mathbb{R}_+)$ and $\Phi(\xi) = \left(1+|\xi|\right)^r$ are given. We define functions Ψ, $\Psi_k (k = 0,1,...,N)$ as $\Psi = \Phi \circ h$, $\Psi_k = A_k \circ h$. Let $M_r(\mathbb{R})$ be some direct complement to linear space $H_r(\mathbb{R})$ in space $L_2(\mathbb{R})$, and $\pi_1 : L_2(\mathbb{R}) \to H_r(\mathbb{R})$, $\pi_2 : L_2(\mathbb{R}) \to M_r(\mathbb{R})$ be operators of projecting linked with $\pi_1 y + \pi_2 y = y$ for $y \in L_2(\mathbb{R})$. Let's define spaces $W_k = \{f; \Psi_k^{-1} f \in L_2(\mathbb{R}_+,\rho)\}$ $(k = 1,2,...,N)$, $W = \{f; \Psi^{-1} f \in L_2(\mathbb{R}_+,\rho)\}$ and operator $\pi_3 : W \to L_2(\mathbb{R}_+,\rho)$ acting identically on $L_2(\mathbb{R}_+,\rho) \bigcap W$. Let S_Γ be operator of singular integration along the contour Γ, i.e. $(S_\Gamma y)(t) = \dfrac{1}{\pi i} v.p. \int\limits_\Gamma y(\tau) \dfrac{d\tau}{\tau - t}$ $(t \in \Gamma)$.

Let's define the functions $\Delta_{k,l}(t) = \exp\left\{i \frac{\lambda_l - \lambda_k}{\alpha} \ln t\right\}$ $(k,l = 1,2,...,N)$. Consider a collection of operators: $\omega_1 : L_2^N(\mathbb{R}_+,\rho) \to H_r(\mathbb{R})$, $\omega_2 : H_r(\mathbb{R}) \to L_2^N(\mathbb{R}_+,\rho)$, $\omega_3 : L_2^N(\mathbb{R}_+,\rho) \to M_r(\mathbb{R})$, $\omega_4 : W_1 \oplus W_2 \oplus \dots \oplus W_N \to L_2^N(\mathbb{R}_+,\rho)$ and $\omega_5 : W_1 \oplus W_2 \oplus \dots \oplus W_N \to L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)$ $\left(\mathbb{R}_- = \mathbb{R} \setminus \mathbb{R}_+\right)$ defined by the following equalities:

$$\omega_1 = \begin{bmatrix} \pi_1 F^{-1}\gamma_1 & \pi_1 F^{-1}\gamma_2 & \cdots & \pi_1 F^{-1}\gamma_N \end{bmatrix},$$

$$\omega_2 = \begin{bmatrix} \gamma_1^{-1} F A_0(D) + S_{\mathbb{R}_+}\gamma_1^{-1} F A_1(D) \\ S_{\mathbb{R}_+}\gamma_2^{-1} F A_2(D) \\ \vdots \\ S_{\mathbb{R}_+}\gamma_N^{-1} F A_N(D) \end{bmatrix}, \quad \omega_3 = \begin{bmatrix} \pi_2 F^{-1}\gamma_1 & \pi_2 F^{-1}\gamma_2 & \cdots & \pi_2 F^{-1}\gamma_N \end{bmatrix},$$

$$\omega_4=\begin{bmatrix}\Lambda_{\Psi_1^{-1}} & 0 & \cdots & 0\\ 0 & \Lambda_{\Psi_2^{-1}} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \Lambda_{\Psi_N^{-1}}\end{bmatrix},\quad \omega_5=\begin{bmatrix}\omega_{11}^5 & \cdots & \omega_{1N}^5\\ \vdots & & \vdots\\ \omega_{N1}^5 & \cdots & \omega_{NN}^5\end{bmatrix},$$

where operators $\omega_{ij}^5 : W_j \to L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)$ $(i,j=1,2,...,N)$ are defined by equalities $(\omega_{ij}^5 v)(t)=\frac{1}{\pi i}\int_0^\infty \tilde{A}_i(\tau)\left(\pi_3\Lambda_\Psi\gamma_0^{-1}\gamma_j\Lambda_{\Psi_j^{-1}}v\right)(\tau)\frac{d\tau}{\tau-t}$, and $\tilde{A}_i=\Delta_{i,0}\Psi_i\Psi^{-1}$ on $\mathbb{R}_+$. Note, that pseudodifferential operator $A(x,D)$ $\left(A(x,D): H_r(\mathbb{R})\to H_r(\mathbb{R})\oplus M_r(\mathbb{R})\right)$ could be represented by the equality

$$A(x,D)=\begin{bmatrix}I_{H_r(\mathbb{R})}+\omega_1\omega_2\\ \omega_3\omega_2\end{bmatrix}.$$

Let's also consider the linear operators

$T_{12}: L_2^N\left(\mathbb{R}_+,\rho\right)\oplus M_r(\mathbb{R})\oplus L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)\to H_r(\mathbb{R})\oplus M_r(\mathbb{R})$,

$T_{21}: H_r(\mathbb{R})\to W_1\oplus W_2\oplus\cdots\oplus W_N\oplus L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)$,

$T_{22}: L_2^N\left(\mathbb{R}_+,\rho\right)\oplus M_r(\mathbb{R})\oplus L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)\to W_1\oplus W_2\oplus\cdots\oplus W_N\oplus L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)$,

$K_{11}: H_r(\mathbb{R})\oplus M_r(\mathbb{R})\to H_r(\mathbb{R})$,

$K_{12}: W_1\oplus W_2\oplus\cdots\oplus W_N\oplus L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)\to H_r(\mathbb{R})$,

$K_{21}: H_r(\mathbb{R})\oplus M_r(\mathbb{R})\to L_2^N\left(\mathbb{R}_+,\rho\right)\oplus M_r(\mathbb{R})\oplus L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)$,

$\tilde{K}: W_1\oplus W_2\oplus\cdots\oplus W_N\oplus L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)\to L_2^N\left(\mathbb{R}_+,\rho\right)\oplus M_r(\mathbb{R})\oplus L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)$,

defined by equalities

$$T_{12}=\begin{bmatrix}-\omega_1 & 0 & 0\\ -\omega_3 & I_{M_r(\mathbb{R})} & 0\end{bmatrix},\quad T_{21}=\begin{bmatrix}-\omega_4^{-1}\omega_2\\ \omega_5\omega_4^{-1}\omega_2\end{bmatrix},\quad T_{22}=\begin{bmatrix}\omega_4^{-1} & 0 & 0\\ -\omega_5\omega_4^{-1} & 0 & I_{L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)}\end{bmatrix},$$

$$K_{11}=\begin{bmatrix}I_{H_r(\mathbb{R})} & 0\end{bmatrix},\quad K_{12}=\begin{bmatrix}\omega_1\omega_4 & 0\end{bmatrix},\quad K_{21}=\begin{bmatrix}\omega_2 & 0\\ 0 & I_{M_r(\mathbb{R})}\\ 0 & 0\end{bmatrix},$$

$$\tilde{K}=\begin{bmatrix}\omega_4+\omega_2\omega_1\omega_4 & 0\\ \omega_3\omega_4 & 0\\ \omega_5 & I_{L_2^N\left(\mathbb{R}_-,\rho(|t|)\right)}\end{bmatrix}.$$

By direct calculation it is easy to verify the equality

$$\begin{bmatrix}A(x,D) & T_{12}\\ T_{21} & T_{22}\end{bmatrix}=\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & \tilde{K}\end{bmatrix}^{-1}.$$

The next lemma follows from the results of the paper [2]:

Lemma 1. Spaces $\mathrm{Ker}A(x,D)$ and $\mathrm{Ker}\tilde{K}$ are isomorphic, besides the following equalities are true $\mathrm{Ker}A(x,D)=K_{12}\mathrm{Ker}\tilde{K}$, $\mathrm{Ker}\tilde{K}=T_{21}\mathrm{Ker}A(x,D)$.

3⁰. Operator $U: W_1\oplus W_2\oplus\cdots\oplus W_N\to W\oplus L_2^{N-1}(\mathbb{R}_+,\rho)$ is defined through the following equality:

$$U=\begin{bmatrix} \Lambda_\Psi\gamma_0^{-1}\gamma_1\Lambda_{\Psi_1^{-1}} & \Lambda_\Psi\gamma_0^{-1}\gamma_2\Lambda_{\Psi_2^{-1}} & \Lambda_\Psi\gamma_0^{-1}\gamma_3\Lambda_{\Psi_3^{-1}} & \cdots & \Lambda_\Psi\gamma_0^{-1}\gamma_N\Lambda_{\Psi_N^{-1}} \\ 0 & \Lambda_{\Psi_2^{-1}} & 0 & \cdots & 0 \\ 0 & 0 & \Lambda_{\Psi_3^{-1}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \Lambda_{\Psi_N^{-1}} \end{bmatrix}.$$

Let's define also the operator $K=\Lambda_M+S_{\mathbb{R}}\Lambda_N : L_2^N(\mathbb{R},\rho(|t|))\to L_2^N(\mathbb{R},\rho(|t|))$, where

$$M=\begin{bmatrix} \Psi^{-1}\Delta_{1,0}+\Psi_0\Psi^{-1}\Delta_{1,0} & -\Delta_{1,2} & -\Delta_{1,3} & \cdots & -\Delta_{1,N} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$N=\begin{bmatrix} \Psi_1\Psi^{-1}\Delta_{1,0} & 0 & \cdots & 0 \\ \Psi_2\Psi^{-1}\Delta_{2,0} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \Psi_N\Psi^{-1}\Delta_{N,0} & 0 & \cdots & 0 \end{bmatrix}$$

on $\mathbb{R}_+$, and correspondingly equal to unit and zero matrices on $\mathbb{R}_-$.

Lemma 2. If vector-function $z=(z_+,z_-)^T$, where $z_+\in W_1\oplus W_2\oplus\cdots\oplus W_N$ and $z_-\in L_2^N(\mathbb{R}_-,\rho(|t|))$, is a solution of the equation $\tilde{K}z=0$, then vector-function u, being equal to Uz_+ on $\mathbb{R}_+$ and to z_- on $\mathbb{R}_-$, is a solution of equation $Ku=0$.

Vice versa, if vector-function u is a solution of the equation $Ku=0$, then vector-function z, being equal to $U^{-1}u$ on $\mathbb{R}_+$ and to u on $\mathbb{R}_-$, is a solution of the equation $\tilde{K}z=0$.

Proof. First of all let's note that for the function f, defined on $\mathbb{R}_+$, the condition $f\in L_2(\mathbb{R}_+,\rho)$ is equivalent to $F^{-1}\gamma_0\Lambda_{\Psi^{-1}}f\in H_r(\mathbb{R})$. Indeed, let $F^{-1}\gamma_0\Lambda_{\Psi^{-1}}f\in H_r(\mathbb{R})$, consequently $\Phi(D)F^{-1}\gamma_0\Lambda_{\Psi^{-1}}f\in L_2(\mathbb{R})$. Hence

$$\gamma_0^{-1}F\Phi(D)F^{-1}\gamma_0\Lambda_{\Psi^{-1}}f=\gamma_0^{-1}\Lambda_\Phi\gamma_0\Lambda_{\Psi^{-1}}f=f\in L_2(\mathbb{R}_+,\rho).$$

Vice versa, let $f\in L_2(\mathbb{R}_+,\rho)$. Consequently $F^{-1}\gamma_0 f\in L_2(\mathbb{R})$, and therefore $\Phi^{-1}(D)F^{-1}\gamma_0 f=F^{-1}\Lambda_{\Phi^{-1}}\gamma_0 f=F^{-1}\gamma_0\Lambda_{\Psi^{-1}}f\in H_r(\mathbb{R})$.

For the vector-function $z=\left(z_{+},z_{-}\right)^{T}$, which is the solution for the equation $\tilde{K}z=0$, the following equalities are true:

$$\omega_4 z_{+}+\omega_2\omega_1\omega_4 z_{+}=0,\quad \omega_3\omega_4 z_{+}=0,\quad \omega_5 z_{+}+z_{-}=0. \tag{1}$$

Let $Uz_{+}=\left(u_1^{+},u_2^{+},...,u_N^{+}\right)^{T}$. Since from the second equality of (1) it follows that $F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}\in H_r\left(\mathbb{R}\right)$, then $u_1^{+}\in L_2\left(\mathbb{R}_{+},\rho\right)$. From the definitions of spaces $W_k (k=2,...,N)$ it follows that $u_k^{+}\in L_2\left(\mathbb{R}_{+},\rho\right)$ $(k=2,...,N)$. Thus $u\in L_2^N\left(\mathbb{R},\rho\left(|t|\right)\right)$. First equality (1) could be represented in the following form:

$$\begin{gathered}\gamma_1^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}+\gamma_1^{-1}FA_0\left(D\right)\pi_1F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}-\gamma_1^{-1}\gamma_2u_2^{+}-\cdots-\gamma_1^{-1}\gamma_Nu_N^{+}+\\+S_{\mathbb{R}_{+}}\gamma_1^{-1}FA_1\left(D\right)\pi_1F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}=0,\\u_k^{+}+S_{\mathbb{R}_{+}}\gamma_k^{-1}FA_k\left(D\right)\pi_1F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}=0\quad (k=2,...,N).\end{gathered}\tag{2}$$

Taking into account that $\pi_1F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}=F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}$, we find

$$\begin{gathered}\gamma_1^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}+\gamma_1^{-1}FA_0\left(D\right)F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}-\gamma_1^{-1}\gamma_2u_2^{+}-\cdots-\gamma_1^{-1}\gamma_Nu_N^{+}+\\+S_{\mathbb{R}_{+}}\gamma_1^{-1}FA_1\left(D\right)F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}=0,\\u_k^{+}+S_{\mathbb{R}_{+}}\gamma_k^{-1}FA_k\left(D\right)F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}=0\quad (k=2,...,N).\end{gathered}\tag{3}$$

Third equality (1) could be represented in the form

$$\frac{1}{\pi i}\int_0^{\infty}\tilde{A}_k(\tau)\left(\pi_3u_1^{+}\right)(\tau)\frac{d\tau}{\tau-t}+z_k^{-}=0\quad\left(k=1,...,N\right). \tag{4}$$

Now taking into account that $\pi_3u_1^{+}=u_1^{+}$, we find

$$\frac{1}{\pi i}\int_0^{\infty}\tilde{A}_k(\tau)u_1^{+}(\tau)\frac{d\tau}{\tau-t}+z_k^{-}=0\quad\left(k=1,...,N\right). \tag{5}$$

The equalities (3) and (5) means that $Ku=0$.

Let now $Ku=0$. Then, if $u_{+}=\left(u_1^{+},u_2^{+},...,u_N^{+}\right)^{T}$ is equal to vector-function u on $\mathbb{R}_{+}$, and $z_{-}=\left(z_1^{-},z_2^{-},...,z_N^{-}\right)^{T}$ is equal to vector-function u on $\mathbb{R}_{-}$, the equality $Ku=0$ is equivalent to (3) and (5). Taking into account that $u_1^{+}\in L_2\left(\mathbb{R}_{+},\rho\right)$, we find that $F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}\in H_r\left(\mathbb{R}\right)$, i.e. $\pi_1F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}=F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}$, $\pi_2F^{-1}\gamma_0\Lambda_{\psi^{-1}}u_1^{+}=0$ and $\pi_3u_1^{+}=u_1^{+}$. From here follows the truth of the equalities (2) and (4). Consequently $z_{+}=U^{-1}u_{+}$ and z_{-} suffice the equalities (1), which means $\tilde{K}z=0$.

From Lemmas 1 and 2 follows

Theorem 1. If $A\left(x,D\right)y=0$, and functions $u_k(x)$ when $x>0$ are defined through the equalities

$$u_1=\Lambda_{\psi}\gamma_0^{-1}Fy,\quad u_k=-S_{\mathbb{R}_{+}}\gamma_k^{-1}FA_k\left(D\right)y\quad (k=2,...,N),$$

then vector-function u, defined through the equality $u=(u_1,u_2,\ldots,u_N)^T$ when $x>0$ and equality $u(x)=-\frac{1}{\pi i}\int_0^{\infty} N(t)u(t)\frac{dt}{t-x}$ when $x<0$, suffice to equation $Ku=0$.

Vice versa, if for the vector-function $u=(u_1,u_2,\ldots,u_N)^T$ $Ku=0$, then function $y(x)=\int_{-\infty}^{\infty} e^{-ixt}e^{st}\Phi_0^{-1}(t)u_1\left(e^{\alpha t}\right)dt$ suffice to equation $A(x,D)y=0$.

4⁰. Due to Theorem 1 the solution of equation $A(x,D)y=0$ in class $H_r(\mathbb{R})$ is brought to description of set $\mathrm{Ker}K$.

Let $P_{\mathbb{R}}^{\pm}=I\pm S_{\mathbb{R}}$ by analytic projectors. Let's assume that $(M\pm N)^{-1}\in L_\infty$. Operator K could be represented in the form $K=P_{\mathbb{R}}^{+}\Lambda_{M+N}+P_{\mathbb{R}}^{-}\Lambda_{M-N}=\left(P_{\mathbb{R}}^{+}\Lambda_G+P_{\mathbb{R}}^{-}\right)\Lambda_{M-N}$, where $G=(M+N)(M-N)^{-1}$. Let $\tilde{\Psi}=\left(1+\Psi_0-\sum_{i=1}^{N}\Psi_i\right)^{-1}$. Through direct calculation it is easy to get convinced that

$$G=\begin{bmatrix} 1+2\Psi_1\tilde{\Psi} & 2\Psi_1\tilde{\Psi}\Delta_{1,2} & 2\Psi_1\tilde{\Psi}\Delta_{1,3} & \cdots & 2\Psi_1\tilde{\Psi}\Delta_{1,N} \\ 2\Psi_2\tilde{\Psi}\Delta_{2,1} & 1+2\Psi_2\tilde{\Psi} & 2\Psi_2\tilde{\Psi}\Delta_{2,3} & \cdots & 2\Psi_2\tilde{\Psi}\Delta_{2,N} \\ 2\Psi_3\tilde{\Psi}\Delta_{3,1} & 2\Psi_3\tilde{\Psi}\Delta_{3,2} & 1+2\Psi_3\tilde{\Psi} & & \\ \vdots & \vdots & & \ddots & \\ 2\Psi_N\tilde{\Psi}\Delta_{N,1} & 2\Psi_N\tilde{\Psi}\Delta_{N,2} & & & 1+2\Psi_N\tilde{\Psi} \end{bmatrix}$$

on $\mathbb{R}_+$, and coincides with the unit matrix on $\mathbb{R}_-$.

Let $L_2^{\pm}(\mathbb{R})=P_{\mathbb{R}}^{\pm}(L_2(\mathbb{R}))$. Under the factorization of the matrix-function G in space $L_2(\mathbb{R},\rho(|t|))$ we understand the representation $G(t)=G_-(t)\Xi(t)G_+(t)$ $(t\in\mathbb{R})$, where

$$\frac{1}{t-i}(t^{\beta/2}G_-(t))^{\pm1}\in L_2^-(\mathbb{R}),\quad \frac{1}{t+i}(t^{-\beta/2}G_+(t))^{\pm1}\in L_2^+(\mathbb{R}),$$
$$\Xi(t)=\operatorname{diag}\left(\left(\frac{t-i}{t+i}\right)^{k_1}\cdots\left(\frac{t-i}{t+i}\right)^{k_N}\right),$$

operator $\Lambda_{G_+^{-1}}S_{\mathbb{R}}\Lambda_{G_+}$ is bounded in $L_2^N(\mathbb{R})$, and $k_1\geq\cdots\geq k_N$ are the integers named particular indices of matrix-function G. Here the belongness to the functional classes is considered element-wise.

Using the standard considerations (see [2]) and general theory of matrix singular operators (see [5–7]), in those cases when matrix-function G allows factorization in $L_2(\mathbb{R},\rho(|t|))$, kernel of operator K allows description in terms of factors of matrix-function G. Precisely, if $k_1 \geq k_2 \geq \cdots \geq k_s \geq 0 > k_{s+1} \geq \cdots \geq k_N$, then Ker$K$ consists from vector-functions of a form $u = (M - N)^{-1} G_+^{-1} p$, where $p = (p_1, ..., p_N)^T$ is a polynomial vector for which the following conditions are true: $p_1 = p_2 = ... = p_s = 0$, and $\deg p_j \leq |k_j| - 1$ $(j = s+1, ..., N)$. Particularly, dim Ker$K$ is equal to the absolute value of sum of the negative particular indices.

As an example let's note that equation $y''(x) + \sum_{k=1}^{N} c_k \varphi(x, \lambda_k) y(x) = 0$ $(c_k \in \mathbb{C})$ is brought to the factorization of the matrix-function

$$G = \begin{bmatrix} 1 + a_1 \tilde{\Psi} & a_1 \tilde{\Psi} \Delta_{1,2} & a_1 \tilde{\Psi} \Delta_{1,3} & \cdots & a_1 \tilde{\Psi} \Delta_{1,N} \\ a_2 \tilde{\Psi} \Delta_{2,1} & 1 + a_2 \tilde{\Psi} & a_2 \tilde{\Psi} \Delta_{2,3} & \cdots & a_2 \tilde{\Psi} \Delta_{2,N} \\ a_3 \tilde{\Psi} \Delta_{3,1} & a_3 \tilde{\Psi} \Delta_{3,2} & 1 + a_3 \tilde{\Psi} & & \\ \vdots & \vdots & & \ddots & \\ a_N \tilde{\Psi} \Delta_{N,1} & a_N \tilde{\Psi} \Delta_{N,2} & & & 1 + a_N \tilde{\Psi} \end{bmatrix}$$

on $\mathbb{R}_+$, and coincides with the unit matrix on $\mathbb{R}_-$, where $a_k = 2c_k$ $(k = 1, 2, ..., N)$, and $\tilde{\Psi}(x) = -\left(\alpha^{-1} \ln x\right)^2 - \sum_{k=1}^{N} c_k$.

5⁰. Let's consider the pseudodifferential operator $A(x, D)$ with a symbol of the form $A(x, \xi) = 1 + P(\varphi(x, \lambda)) A_0(\xi)$, where $P(z) = \prod_{k=1}^{N} (z - \nu_k)$ is an arbitrary polynom. Let's define operators

$\tilde{\omega}_1 = \pi_1 F^{-1} \gamma : L_2(\mathbb{R}_+, \rho) \to H_r(\mathbb{R})$, $\tilde{\omega}_2 = \sum_{k=0}^{N} c_k S_{\mathbb{R}_+}^k \gamma^{-1} F A_0(D) : H_r(\mathbb{R}) \to L_2(\mathbb{R}_+, \rho)$,

$\tilde{\omega}_3 = \pi_2 F^{-1} \gamma : L_2(\mathbb{R}_+, \rho) \to M_r(\mathbb{R})$, $\tilde{\omega}_4 = \Lambda_{\psi^{-1}} : \widetilde{W} \to L_2(\mathbb{R}_+, \rho)$,

where $\widetilde{W} = \left\{ f : A_0^{-1}(h) f \in L_2(\mathbb{R}_+, \rho) \right\}$.

By direct calculation, we get the equality

$$\begin{bmatrix} I_{H_r(\mathbb{R})} + \tilde{\omega}_1 \tilde{\omega}_2 & -\tilde{\omega}_1 & 0 \\ \tilde{\omega}_3 \tilde{\omega}_2 & -\tilde{\omega}_3 & I_{M_r(\mathbb{R})} \\ -\tilde{\omega}_4^{-1} \tilde{\omega}_2 & \tilde{\omega}_4^{-1} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} I_{H_r(\mathbb{R})} & 0 & \tilde{\omega}_1 \tilde{\omega}_4 \\ \tilde{\omega}_2 & 0 & \tilde{\omega}_4 + \tilde{\omega}_2 \tilde{\omega}_1 \tilde{\omega}_4 \\ 0 & I_{M_r(\mathbb{R})} & \tilde{\omega}_3 \tilde{\omega}_4 \end{bmatrix}.$$

Using this matrix identity one can prove a result similar to Theorem 1, where operator K plays role of the singular operator, defined through the equality

$$Kz = A_0^{-1}(h) z + \sum_{k=0}^{N} c_k S_{\mathbb{R}_+}^k z.$$

Following § 3.1 of [1], the investigation of the pseudodifferential equation $A(x, D) y = 0$ is brought to the factorization of the matrix-function $G = G_+^{-1} G_-$, where

$$G_{\pm}=\begin{bmatrix} 1 & (\pm 1-\nu_1)A(h) & 0 & \cdots & 0 & 0 \\ 0 & 1 & \nu_2 \mp 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & \nu_{N-1}\mp 1 \\ \nu_N \mp 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

when $x>0$, and $G_{\pm}$ are equal to unit matrix when $x<0$.

Particularly, the solution of the differential equation $y''(x)+\left(\varphi(x,\lambda)-\nu_1\right)\left(\varphi(x,\lambda)-\nu_2\right)y(x)=0$ from class $H_r(\mathbb{R})$ could be expressed by factors of matrix

$$G=\begin{bmatrix} \dfrac{1+A_0(h)(-1+\nu_1)(1+\nu_2)}{1+A_0(h)(-1+\nu_1)(-1+\nu_2)} & -\dfrac{2A_0(h)}{1+A_0(h)(-1+\nu_1)(-1+\nu_2)} \\ \dfrac{2}{1+A_0(h)(-1+\nu_1)(-1+\nu_2)} & \dfrac{1+A_0(h)(1+\nu_1)(-1+\nu_2)}{1+A_0(h)(-1+\nu_1)(-1+\nu_2)} \end{bmatrix}$$

on $\mathbb{R}_+$, and coincides with the unit matrix on $\mathbb{R}_-$.

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Միաչափ պսևդոդիֆերենցիալ օպերատորների մի դասի և
սինգուլյար ինտեգրալ օպերատորների կապի մասին

Աշխատանքում դիտարկվում է $A(x,\xi)=A_0(\xi)+\sum_{k=1}^{N}\text{th}\frac{\pi}{\alpha}\left(x-\lambda_k+i\frac{\alpha\beta}{2}\right)A_k(\xi)$ ($x,\xi,\lambda_k\in\mathbb{R}$, $\alpha>0$, $-1<\beta<1$, $k=1,2,\ldots,N$) տեսքի սիմվոլով համասեռ միաչափ պսևդոդիֆերեն-ցիալ հավասարում, որտեղ $A_k(\xi)$-երը $(k=0,1,\ldots,N)$ լոկալ հանրագումարելի ֆունկցիա-ներ են՝ r ոչ բացասական կարգի սիմվոլների դասից:

Առաջարկվում է պսևդոդիֆերենցիալ հավասարումը Կոշու կորիզով միաչափ սինգուլյար ինտեգրալ հավասարումների համակարգի հանգեցնելու մեթոդ:

О связи одного класса одномерных псевдодифференциальных операторов с
сингулярными интегральными операторами

В работе рассматривается однородное одномерное псевдодифференциальное уравнение с символом вида $A(x,\xi)=A_0(\xi)+\sum_{k=1}^{N}\text{th}\frac{\pi}{\alpha}\left(x-\lambda_k+i\frac{\alpha\beta}{2}\right)A_k(\xi)$, ($x,\xi,\lambda_k\in\mathbb{R}$, $\alpha>0$, $-1<\beta<1$, $k=1,2,\ldots,N$), где $A_k(\xi)$ $(k=0,1,\ldots,N)$ – локально суммируемые функции из класса символов неотрицательного порядка *r*.

Предлагается метод сведения псевдодифференциального уравнения к системе одномерных сингулярных интегральных уравнений с ядром Коши.