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Mathematics

PLEIJEL TYPE IDENTITIES

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In the present paper generalizations of classical Pleijel identities are obtained. We refer these identities as Pleijel type identities. Particular cases of these identities are proved in [1], [3] and [5].

Keywords**:** bounded convex domains, combinatorial decompositions, combinatorial algorithm.

Let **D** be a bounded convex domain in the plane $\mathbf{R}^2$ with piecewise-smooth boundary $\partial\mathbf{D}$. We also assume that $\partial\mathbf{D}$ contains no line segments. Let be a finite non-degenerate set **(**the points can be inside the domain, as well as outside of **D**).

Let us fix the directed lines $g_1, g_2, ..., g_n$ that intersect the domain **D**. These lines generate chords $g_i \cap$ **D**, $i = 1, ..., n$, that we denote by $\chi(g_1), \chi(g_2), ..., \chi(g_n)$. The set $\{P_i\}$ consists of the endpoints of the above mentioned chords, lying on $\partial\mathbf{D}$. Hence**,** $\{P_i\}_{i=1}^{2n} \subset \partial\mathbf{D}$ consists of $2n$ points. Denote by ρ_{ij} the segment with the endpoints P_i and P_j, while $|\rho_{ij}|$ is its length. We set

$$[\rho_{ij}] = \{g \in \mathbf{G}\colon\ g \cap \rho_{ij} \neq \varphi\},$$

where **G** is the space of directed lines in the plane.

Let Br$\{P_i\}$ be the minimal (finite) ring of subsets of **G** generated by all sets $[\rho_{ij}]$. Let $B = \{g \in \mathbf{G}\colon\ g$ intersects all $\chi(g_i), i = 1, ..., n\} = \bigcap_{i=1}^{n} [\chi(g_i)]$, and A be an element of algebra a ($a\{Q_i\}$ is the minimal algebra of **G** containing all sets $[\rho_{ij}]$). It is easy to see that

$$A \cap B \in r(\{P_i\} \cup \{Q_i\}).$$

Using R.V. Ambartzumian's combinatorial formula (see [1] or [2]) for $\mu(A \cap B)$, where μ is the measure invariant with respect to all Euclidean motions in the space **G**, we get

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$$\mu(A\cap B)=\sum_{(P_i,P_j)}\rho_{ij}c_{ij}(B)I_A(g_{ij})+\sum_{(Q_i,Q_j)}\rho_{ij}c_{ij}(A)I_B(g_{ij})+ \\ +\sum_{(P_i,Q_j)}\rho_{ij}(I_B(i^+,j)-I_B(i^-,j))(I_A(i,j^-)-I_A(i,j^+)). \tag{1.1}$$

In (1.1) the sums are taken over all ordered pairs of points (in particular, in the last sum along with the term P_iQ_j we have term Q_jP_i). As usual, $I_A(g)$ is the indicator of the set A. The algorithm for calculating coefficients c_{ij} is given in [1–3].

In the space $(g_1,g_2,...,g_n)\in \mathbf{G}^n$ we consider the measure

$$d\mu^{(n)}=dg_1\cdots dg_n,$$

where each dg_i coincides with the element of invariant measure μ in the space $\mathbf{G}$.

We integrate (1.1) by the measure $d\mu^{(n)}$. Note, that (1.1) is valid for almost all sequences of chords $\chi(g_1),\chi(g_2),...,\chi(g_n)$, because $\partial\mathbf{D}$ contains no line segments. Therefore, for almost all $\chi(g_1),\chi(g_2),...,\chi(g_n)$ the set $\{P_i\}\cup\{Q_i\}$ is non-degenerate.

The Main Result. Integrating the left-hand side of (1.1), we obtain

$$\int_{\mathbf{G}^n}\mu(A\cap B)d\mu^{(n)}=\int_{\mathbf{G}^n}I_{A\cap B}(g)dg=\int_{[\mathbf{D}]}I_A(g)dg\int_{\mathbf{G}^n}I_B(g)d\mu^{(n)}=\int I_A(g)(4\chi(g))^n dg.$$

Here we used

$$\int_{[\chi(g)]}dg_i=4\chi(g). \tag{2.1}$$

Now we calculate the integrals for the sums in the right–hand side of (1.1). We start with the second term, which can be easily calculated using formula (2.1). We have

$$\int_{\mathbf{G}^n}d\mu^{(n)}\sum_{(Q_iQ_j)}\rho_{ij}c_{ij}(A)I_B(g_{ij})=\sum_{(Q_iQ_j)}\rho_{ij}c_{ij}(A)(4\chi_{ij})^n, \tag{2.2}$$

where $\chi_{ij}=\chi(g_{ij})$ is the length of the chord of domain **D,** passing through the points Q_i and Q_j. If points Q_i and $Q_j\notin\mathbf{D}$, then possibly the line g_{ij} does not intersect $\mathbf{D}$ and, therefore, $\chi_{ij}=0$.

Let us integrate the first term in the right-hand side of (1.1). Arguing as in [2], pages 156–157, and [4], we get by symmetry

$$\int_{\mathbf{G}^n}d\mu^{(n)}\sum_{(P_iP_j)}\rho_{ij}c_{ij}(B)I_A(g_{ij})=n\int_A(4\chi)^n dg-4n(n-1)\iint_{(\partial\mathbf{D})^2}(4\chi_{12})^{n-1}\cos\alpha_1\cos\alpha_2 I_A(g_{12})dl_1dl_2,$$

where g_{12} is the directed line joining the points l_1 and l_2 of $\partial\mathbf{D}$, while α_1 and α_2 are the interior angles between χ_{12} and $\partial\mathbf{D}$ at the points l_1 and l_2, correspondingly, that lie in the same half-plane with respect to g_{12}.

Let us integrate the last sum in (1.1). Let l be the coordinate of the point on $\partial\mathbf{D}$, from which the chord $\chi(g_1)$ emerges. For each point Q_j the directed line from Q_j to l is denoted by g_{jl}, $\chi_{jl}=\chi(g_{jl})$, while ρ_{jl} is the distance between Q_j and l. Denote by β_{jl} (β_{lj}) the right interior angle of g_{jl} (g_{lj}) with $\partial\mathbf{D}$ at l. Reasoning by analogy to [1] and [3], we get

$$\int_{\mathbf{G}^n} d\mu^{(n)} \sum_{(P_i,Q_j)} \rho_{ij}(I_B(i^+,j)-I_B(i^-,j))(I_A(i,j^-)-I_A(i,j^+)) =$$
$$= 4n\sum_{Q_j}\int_{\partial\mathbf{D}} (4\chi_{jl})^{n-1}\rho_{jl}[I_A(j^-,l)-I_A(j^+,l)]\cos\beta_{jl}dl +$$
$$+ 4n\sum_{Q_j}\int_{\partial\mathbf{D}} (4\chi_{jl})^{n-1}\rho_{jl}[I_A(l,j^-)-I_A(l,j^+)]\cos\beta_{lj}dl.$$

Let us assume that l increases in the clockwise direction around $\partial\mathbf{D}$. Then

$$\rho_{jl}\cos\beta_{jl}\,dl = -\frac{1}{2}d\rho_{jl}^2, \quad \rho_{jl}\cos\beta_{lj}\,dl = \frac{1}{2}d\rho_{jl}^2. \tag{2.3}$$

Substituting (2.3) into the previous formula and integrating by parts, we obtain

$$-\frac{1}{2}\sum_s\{(4\chi_{jl})^{n-1}[I_A(j^-,l)-I_A(j^+,l)]\rho_{jl}^2\}\Big|_{l_{js}+0}^{l_{js}-0} + 2(n-1)\int_{\partial\mathbf{D}}(4\chi_{jl})^{n-2}[I_A(j^-,l)-I_A(j^+,l)]\rho_{jl}^2 d\chi_{jl} \tag{2.4}$$

and

$$\frac{1}{2}\sum_s\{(4\chi_{jl})^{n-1}[I_A(l,j^-)-I_A(l,j^+)]\rho_{jl}^2\}\Big|_{l_{js}+0}^{l_{js}-0} - 2(n-1)\int_{\partial\mathbf{D}}(4\chi_{jl})^{n-2}[I_A(l,j^-)-I_A(l,j^+)]\rho_{jl}^2 d\chi_{jl}, \tag{2.5}$$

here for fixed j, s enumerates the set $\{l_{js} : j \text{ fixed}\}$ of points of discontinuity of the expressions in square brackets.

Consider two cases.

1) *The case* $Q_j \in \mathbf{D}$. We make the change of variable $l \to l^*$ in (2.5), where l^* denotes the point other than l, where g_{jl} meets $\partial\mathbf{D}$. Using the relationship $I_A(j^\pm,l) = I_A(l^*,j^\pm)$, the integral in (2.5) can be written as

$$-2(n-1)\int_{\partial\mathbf{D}}(4\chi_{jl})^{n-2}[I_A(j^-,l)-I_A(j^+,l)]\rho_{jl*}^2 d\chi_{jl}.$$

Since $\rho_{jl}+\rho_{jl*}=\chi_{jl}$, the sum of the integrals (2.4) and (2.5) is equal to

$$\frac{1}{2}(n-1)\int_{\partial\mathbf{D}}(4\chi_{jl})^{n-2}[I_A(j^-,l)-I_A(j^+,l)](\rho_{jl}-\rho_{jl*})d\chi_{jl}.$$

Thus, the total contribution of the integral terms in (2.4) and (2.5) is

$$4n(n-1)\sum_{Q_j}\int_{\partial\mathbf{D}}(4\chi_{jl})^{n-1}[I_A(j^-,l)-I_A(j^+,l)]\rho_{jl}d\chi_{jl}. \tag{2.6}$$

2) *The case* $Q_j \notin \mathbf{D}$ (the point Q_j lies outside the domain **D**). In this case the total contribution of the integral terms in (2.4) and (2.5) takes the form

$$\int_{\partial\mathbf{D}} f_j(l)\rho_{jl}^2\,d\chi_{jl} = A,$$

$$\int_{\partial\mathbf{D}} f_j(l)\rho_{jl*}^2\,d\chi_{jl*} = A,$$

where $f_j(l) = 2(n-1)(4\chi_{jl})^{n-2}[I_A(j^-,l) - I_A(j^+,l) - I_A(l,j^-) + I_A(l,j^+)]$.

We made the change of variable $l \to l^*$ in A and applied relationships $I_A(j^\pm,l^*) = I_A(j^\pm,l)$, $I_A(l^*,j^\pm) = I_A(l,j^\pm)$. Summing up the expressions obtained for A, we get

$$2A = \int_{\partial \mathbf{D}} f_j(l)\rho_{jl}^2\, d\chi_{jl} + \int_{\partial \mathbf{D}} f_j(l)\rho_{jl^*}^2\, d\chi_{jl^*}.$$

Let us draw the tangents to the domain **D** from the point Q_j and, thus, divide the boundary $\partial \mathbf{D}$ into two parts $I_1 \cup I_2$. I_1 is the part of $\partial \mathbf{D}$ between the tangency points facing the point Q_j. I_2 is the complement to I_1, $I_2 = \partial \mathbf{D} \setminus I_1$. If $l \in I_1$, then $l^* \in I_2$, and hence $\chi_{jl} = \rho_{jl^*} - \rho_{jl} = \chi_{jl^*}$. Similarly, if $l \in I_2$, then $l^* \in I_1$, and hence $\chi_{jl} = \rho_{jl} - \rho_{jl^*} = \chi_{jl^*}$. We also have

$$d\chi_{jl} = \chi'_{jl}dl, \quad d\chi_{jl^*} = \chi'_{jl^*}dl^* \;\Rightarrow d\chi_{jl^*} = -d\chi_{jl}, \quad dl^* = -dl.$$

Therefore,

$$2A = \int_{I_1 \cup I_2} f_j(l)\rho_{jl}^2\, d\chi_{jl} - \int_{I_1 \cup I_2} f_j(l)\rho_{jl^*}^2\, d\chi_{jl^*} = \int_{I_1} f_j(l)\rho_{jl}^2\, d\chi_{jl} + \int_{I_2} f_j(l)\rho_{jl}^2\, d\chi_{jl} -$$
$$- \int_{I_1} f_j(l)\rho_{jl^*}^2\, d\chi_{jl^*} - \int_{I_2} f_j(l)\rho_{jl^*}^2\, d\chi_{jl^*} = \int_{I_1} f_j(l)(\rho_{jl}^2 - \rho_{jl^*}^2)\, d\chi_{jl} + \int_{I_2} f_j(l)(\rho_{jl}^2 - \rho_{jl^*}^2)\, d\chi_{jl} =$$
$$= -\int_{I_1} f_j(l)\chi_{jl}(\rho_{jl} + \rho_{jl^*})\, d\chi_{jl} + \int_{I_2} f_j(l)\chi_{jl}(\rho_{jl} + \rho_{jl^*}^2)\, d\chi_{jl} = -2\int_{I_1} f_j(l)\chi_{jl}\rho_{jl}\, d\chi_{jl} +$$
$$+ 2\int_{I_2} f_j(l)\chi_{jl}\rho_{jl}\, d\chi_{jl} = 2\int_{\partial \mathbf{D}} \nu_j(l) f_j(l)\chi_{jl}\rho_{jl}\, d\chi_{jl},$$

where

$$\nu_j(l) = \begin{cases} -1, & \text{if } \; l \in I_1, \\ 1, & \text{if } \; l \in I_2. \end{cases}$$

Thus, for integral terms of (2.4) and (2.5) we obtain

$$n\sum_{Q_j} \int_{\partial \mathbf{D}} \nu_j(l) f_j(l)\rho_{jl}\chi_{jl}\, d\chi_{jl}. \tag{2.7}$$

Now let us evaluate the total contribution to the non-integral terms in (2.4) and (2.5). Consider the following four cases.

1) Q_j and Q_k are interior points of **D**. We have (see [2]):

$$\begin{aligned} 4n\sum_{(j,k)} \frac{1}{2}(4\chi_{jk})^{n-1} c_{jk}(A)[\delta_{kj}^2 + \delta_{jk}^2 - (\rho_{jk} + \delta_{kj})^2 - (\rho_{kj} + \delta_{jk})^2] = \\ = -4n\sum_{(j,k)} (4\chi_{jk})^{n-1} c_{jk}(A)\rho_{jk}\chi_{jk} = -n\sum_{(j,k)} (4\chi_{jk})^{n} c_{jk}(A)\rho_{jk}, \end{aligned} \tag{2.8}$$

where $\chi_{jk} = \rho_{jk} + \delta_{jk} + \delta_{kj}$.

2) $Q_j \notin \mathbf{D}$, $Q_k \notin \mathbf{D}$ and lie on different sides with respect to **D**. We have

$$4n\sum_{(j,k)}\frac{1}{2}(4\chi_{jk})^{n-1}c_{jk}(A)[d_j^2+d_k^2-(\rho_{jk}-d_k)^2-(\rho_{kj}-d_j)^2]=$$
$$=-4n\sum_{(j,k)}(4\chi_{jk})^{n-1}c_{jk}(A)(\rho_{jk}-d_j-d_k)\rho_{jk}=-n\sum_{(j,k)}(4\chi_{jk})^{n}c_{jk}(A)\rho_{jk}, \tag{2.9}$$

where $\chi_{jk}=\rho_{jk}-d_j-d_k$.

3) $Q_j\notin\mathbf{D}$, $Q_k\notin\mathbf{D}$ and lie on the same side with respect to **D**. We have

$$4n\sum_{(j,k)}\frac{1}{2}(4\chi_{jk})^{n-1}c_{jk}(A)[(\rho_{jk}+d_k)^2-(\rho_{jk}+d_k+\chi_{jk})^2+(d_k+\chi_{jk})^2-d_k^2]=$$
$$=-4n\sum_{(j,k)}(4\chi_{jk})^{n-1}c_{jk}(A)\rho_{jk}\chi_{jk}=-n\sum_{(j,k)}(4\chi_{jk})^{n}c_{jk}(A)\rho_{jk}, \tag{2.10}$$

where $d_j-d_k=\rho_{jk}$.

4) The point Q_j is outside of domain **D,** and Q_k is interior point for domain **D**. We have

$$4n\sum_{(j,k)}\frac{1}{2}(4\chi_{jk})^{n-1}c_{jk}(A)[d_j^2-(\chi_{jk}+d_j)^2-(\chi_{jk}-\rho_{jk}+d_j)^2-(\rho_{jk}-d_j)^2]=$$
$$=-4n\sum_{(j,k)}(4\chi_{jk})^{n-1}c_{jk}(A)\rho_{jk}\chi_{jk}=-n\sum_{(j,k)}(4\chi_{jk})^{n}c_{jk}(A)\rho_{jk}. \tag{2.11}$$

Thus we finally get the following identity that is a generalization of the classical Pleijel identity:

$$\int_A\chi^n dg=n\iint_{(\partial\mathbf{D})^2}\chi^{n-1}\cos\alpha_1\cos\alpha_2\, I_A(g_{12})dl_1dl_2+\sum_{(Q_i,Q_j)}c_{ij}(A)\rho_{ij}\chi_{ij}^n+$$
$$+n\sum_{Q_j}\left(\int_{\partial\mathbf{D}}I_{\mathbf{D}}(Q_j)\chi_{jl}^{n-1}[I_A(j^+,l)-I_A(j^-,l)]\rho_{jl}d\chi_{jl}+\right. \tag{2.12}$$
$$\left.+\frac{1}{2}\nu_j(l)[1-I_{\mathbf{D}}(Q_j)]\chi_{jl}^{n-1}[I_A(j^+,l)-I_A(j^-,l)-[I_A(l,j^+)-I_A(l,j^-)]\rho_{jl}d\chi_{jl}\right),$$

where

$$I_{\mathbf{D}}(Q_j)=\begin{cases}1, & \text{if } Q_j\in\mathbf{D},\\ 0, & \text{if } Q_j\notin\mathbf{D}.\end{cases}$$

Using the linearity property of (2.12), one can obtain the following relationship for any function f with continuous derivative, satisfying $f(0)=0$:

$$\int_A f(\chi)dg=n\iint_{(\partial\mathbf{D})^2}f'(\chi)\cos\alpha_1\cos\alpha_2\, I_A(g_{12})dl_1dl_2+\sum_{(Q_i,Q_j)}c_{ij}(A)\rho_{ij}f(\chi_{ij})+$$
$$+\sum_{Q_j}\left(\int_{\partial\mathbf{D}}I_{\mathbf{D}}(Q_j)f(\chi_{jl})[I_A(j^+,l)-I_A(j^-,l)]\rho_{jl}d\chi_{jl}+\right. \tag{2.13}$$
$$\left.+\frac{1}{2}\nu_j(l)[1-I_{\mathbf{D}}(Q_j)]f(\chi_{jl})[I_A(j^+,l)-I_A(j^-,l)-[I_A(l,j^+)-I_A(l,j^-)]\rho_{jl}d\chi_{jl}\right).$$

Substituting $A=\mathbf{G}$ in (2.13)**,** we obtain the classical Pleijel identity (see [1–3, 5]). The second particular case is obtained assuming that all points Q_j lie inside the domain **D**. In this case (2.13) has the form of identity (1.14) from [3] (see also (8.14) from [1, 6]). Finally, if A coincides with the set of lines intersecting a segment, lying outside of domain **D**, then identity (2.13) coincides with identity (2.8) from [7].

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Փլեյելի տիպի նույնություններ

Աշխատանքում ստացված են Փլեյելի դասական նույնության ընդհանրացում-ները, որոնք անվանում ենք Փլեյելի տիպի նույնություններ: Այդ նույնությունների մասնավոր դեպքերը դուրս են բերված [1], [3] և [5] աշխատանքներում:

Тождества типа Плейеля

В работе получены обобщения классических тождеств Плейеля, которые названы тождествами типа Плейеля. Частные случаи этих тождеств получены в работах [1], [3] и [5].