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Mathematics

ON SOME FORMULAS FOR THE INDEX OF LINEAR BOUNDED OPERATOR

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Dedicated to the memory of Prof. V. B. Lidskii

We consider the linear bounded operator in infinite dimensional separable Hilbert space satisfying some conditions. We prove formulas that can be used to calculate the index of this operator.

Keywords**:** operator, index, trace, absolute norm.

Let V be a linear bounded operator, acting in an infinite dimensional Hilbert space H, V^* is the adjoint operator, $\ker V$ and $\ker V^*$ are their null-spaces, $\dim(\ker V)$ and $\dim(\ker V^*)$ are dimensions of corresponding subspaces. The index of the operator V is the number

$$\operatorname{ind} V = \dim(\ker V) - \dim(\ker V^*), \tag{1}$$

assuming that these dimensions are finite.

Under some conditions on V we prove some formulas, which can be used to calculate $\operatorname{ind} V$. The conditions on V and the received formulas for $\operatorname{ind} V$ differ from the known ones (see [1]).

Lemma. Let I be the identity operator, and $\lambda \neq 0$ be some number. Then

$$\ker(V^*V) = \ker V, \quad \ker(VV^*) = \ker V^*, \tag{2}$$

$$\ker(V^*V - \lambda I) \cap \ker V = \{0\}, \quad \ker(VV^* - \lambda I) \cap \ker V^* = \{0\}, \tag{3}$$

$$V(\ker(V^*V - \lambda I)) = \ker(VV^* - \lambda I), \ V^*(\ker(VV^* - \lambda I)) = \ker(V^*V - \lambda I), \tag{4}$$

$$\dim(\ker(V^*V - \lambda I)) = \dim(\ker(VV^* - \lambda I)). \tag{5}$$

Proof. If $x \in \ker(V^*V)$, then $(Vx, Vx) = (V^*Vx, x) = 0$. Therefore, $Vx = 0$. It follows that $x \in \ker V$ and $\ker(V^*V) \subset \ker V$. From this and from obvious inclusion $\ker V \subset \ker(V^*V)$ we get the first equality of (2). The second equality of (2) can be proved in the same way. Let $x \in \ker V$ and $x \neq 0$. Then $V^*Vx - \lambda x = -\lambda x \neq 0$, i.e. $x \notin \ker(V^*V - \lambda I)$. This implies the first equality of (3).

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The second equality of (3) can be proved in the same way. If $y\in\ker(V^*V-\lambda I)$, then for $z=Vy$ we have $VV^*z-\lambda z=V(V^*Vy-\lambda y)=0$, i.e. $z\in\ker(VV^*-\lambda I)$. Thus, $V(\ker(V^*V-\lambda I))\subset\ker(VV^*-\lambda I)$. Let $x\in\ker(VV^*-\lambda I)$ and $y=\frac{1}{\lambda}V^*x$. Then $x=Vy$ and $V^*Vy-\lambda y=\frac{1}{\lambda}V^*(VV^*x-\lambda x)=0$, i.e. $y\in\ker(V^*V-\lambda I)$ and $x\in V(\ker(V^*V-\lambda I))$. Therefore, $\ker(VV^*-\lambda I)\subset V(\ker(V^*V-\lambda I))$. From these inclusions we obtain the first equality of (4). The second equality in (4) can be proved by similar reasoning. If $x\in\ker(V^*V-\lambda I)$, then $(Vx,Vy)=\lambda(x,y)$ for all $y\in H$, and if $x\in\ker(VV^*-\lambda I)$, then $(V^*x,V^*y)=\lambda(x,y)$. Hence, using (3) and (4) we obtain that the operator V transforms any orthogonal non-zero system of elements from $\ker(V^*V-\lambda I)$ into an orthogonal non-zero system of elements from $\ker(VV^*-\lambda I)$, and V^* performs this transformation in reversed order. Consequently, (5) holds.

Theorem. Let for some number $c\neq 0$ operators $A=V^*V-cI$ and $B=VV^*-cI$ be compact. Then $c>0$, and if the number $-c$ is an eigenvalue of the multiplicity κ' for A and of the multiplicity κ'' for B (we do not exclude cases $\kappa'=0$ or $\kappa''=0$), then

$$\operatorname{ind}V=\kappa'-\kappa''. \tag{6}$$

Moreover, if the space H is separable and one of the operators A and B belongs to the Hilbert–Schmidt class, then the other one also belongs to the same class, and for their absolute norms $N(A)$ and $N(B)$ the equality

$$\operatorname{ind}V=\frac{1}{c^2}\{N^2(A)-N^2(B)\} \tag{7}$$

holds. If one of the operators A or B belongs to the trace class, then the other one has the same property, and for their traces $\operatorname{sp}A$ and $\operatorname{sp}B$ the equality

$$\operatorname{ind}V=\frac{1}{c}\{\operatorname{sp}B-\operatorname{sp}A\}=\frac{1}{c}\operatorname{sp}(B-A)=\frac{1}{c}\operatorname{sp}(VV^*-V^*V) \tag{8}$$

holds.

Proof. The spectrum $\sigma(A)$ of any compact operator A is at most a countable and bounded set containing zero, and any non-zero element of this set is an eigenvalue of finite multiplicity. Moreover, if the set $\sigma(A)$ is infinite, then zero is the only limiting point of $\sigma(A)$. Evidently the spectrum of the operator V^*V is the set $\sigma(V^*V)=\{\lambda+c:\lambda\in\sigma(A)\}$. Thus $c\in\sigma(V^*V)$. Since V^*V is a non-negative self-adjoint operator, then $c>0$, and A is self-adjoint. Similar statements are true for operators B and VV^*. Particularly $\sigma(VV^*)=\{\lambda+c:\lambda\in\sigma(B)\}$. From this and the statement (5) it follows that $\sigma(A)\setminus\{-c\}=\sigma(B)\setminus\{-c\}$, and if the number $\lambda\neq -c$ is an eigenvalue for one of the operators A and B, then λ is an eigenvalue of the same multiplicity for the other one (in the case $\lambda\neq 0$ this multiplicity is finite). Let $-c$ be an eigenvalue of multiplicity κ' for A and of multiplicity κ'' for B. These multiplicities evidently are finite and

$$\kappa' = \dim(\ker(V^*V)), \quad \kappa'' = \dim(\ker(VV^*)).$$

From here by (1) and (2) we get (6). Put $\sigma = \sigma(A) \setminus \{-c, 0\} = \sigma(B) \setminus \{-c, 0\}$. Any number $\lambda \in \sigma$ is an eigenvalue of the same multiplicity $\kappa(\lambda)$ for both operators A and B. Let A be a Hilbert–Schmidt class operator, i. e. have finite absolute norm $N(A)$ (see [2], pp. 96–103, 208–212). Since the operator A is self-adjoint, then (see [2], p. 209)

$$N^2(A) = c^2\kappa' + \sum_{\lambda \in \sigma} \lambda^2 \kappa(\lambda).$$

Hence the absolute norm $N(B)$ of the operator B is also finite and

$$N^2(B) = c^2\kappa'' + \sum_{\lambda \in \sigma} \lambda^2 \kappa(\lambda).$$

Thus $N^2(A) - N^2(B) = c^2(\kappa' - \kappa'')$. From this and relation (6) we get (7).

Let the operator A belongs to the trace class (see [2], p. 208–212), i. e.

$$\sum_{\lambda \in \sigma} |\lambda| \kappa(\lambda) < \infty.$$

Then the operator B also belongs to the same class. According to the Theorem of V.B. Lidskii (see [2], p. 212; [3], p. 131; [4]), for $\operatorname{sp} A$ and $\operatorname{sp} B$ the following equalities

$$\operatorname{sp} A = -c\kappa' + \sum_{\lambda \in \sigma} \lambda \kappa(\lambda), \quad \operatorname{sp} B = -c\kappa'' + \sum_{\lambda \in \sigma} \lambda \kappa(\lambda)$$

are true. Hence $\operatorname{sp} B - \operatorname{sp} A = c(\kappa' - \kappa'')$ and by (6) we get (8).

The Theorem is proved.

Consider in the space $L^2(a,b)$ with finite or infinite interval (a,b) the following integral operator K:

$$(Kx)(\xi) = \int_a^b K(\xi, \eta) x(\eta) d\eta, \quad x \in L^2(a,b), \ \xi \in (a,b),$$

where the function $K(\xi, \eta)$ satisfies the following condition:

$$\int_a^b \int_a^b |K(\xi, \eta)|^2 d\eta \, d\xi < \infty.$$

It is known (see [2], p. 101–102), that the operator K belongs to the Hilbert–Schmidt class, and its absolute norm $N(K)$ is equal to

$$N^2(K) = \int_a^b \int_a^b |K(\xi, \eta)|^2 d\eta \, d\xi.$$

If the operator K is self-adjoint, then $K(\xi, \eta) = \overline{K(\eta, \xi)}$. For the sake of definiteness we consider the case, where the self-adjoint compact operator K has an infinite set of eigenvalues. Enumerate non-zero eigenvalues λ_n $(n = 1, 2, \ldots)$ in order of decreasing module: $|\lambda_1| \geq |\lambda_2| \geq \cdots$, repeating each eigenvalue according to its multiplicity. Denote by φ_n $(n = 1, 2, \ldots)$ the orthonormal set of corresponding eigenfunctions: $K\varphi_n = \lambda_n \varphi_n$. It is known (see [2], pp. 102, 209), that

$$N^2(K)=\sum_{n=1}^{\infty}\lambda_n^2,$$

$$K(\xi,\eta)=\sum_{n=1}^{\infty}\lambda_n\,\varphi_n(\xi)\overline{\varphi_n(\eta)}, \tag{9}$$

and the functional series in (10) converges in the space $L^2((a,b)\times(a,b))$.

Let the self-adjoint operator K belong to the trace class. Then

$$\sum_{n=1}^{\infty}|\lambda_n|<\infty, \quad \operatorname{sp}K=\sum_{n=1}^{\infty}\lambda_n.$$

We extend each function $x\in L^2(a,b)$ onto $R=(-\infty,\infty)$, putting $x(\xi)=0$ for $\xi\notin(a,b)$. We extend also the function $K(\xi,\eta)$ onto R^2, putting $K(\xi,\eta)=0$ for $(\xi,\eta)\notin(a,b)\times(a,b)$. By (9) we have

$$K(\xi+t,\xi)=\sum_{n=1}^{\infty}\lambda_n\,\varphi_n(\xi+t)\overline{\varphi_n(\xi)}, \tag{10}$$

and the functional series on variables ξ and t converges in the space $L^2(R^2)$. Indeed, this fact follows from the equality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sum_{n=p}^{m}\lambda_n\,\varphi_n(\xi+t)\overline{\varphi_n(\xi)}\right|^2 dt\,d\xi=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sum_{n=p}^{m}\lambda_n\,\varphi_n(\eta)\overline{\varphi_n(\xi)}\right|^2 d\eta\,d\xi=$$

$$=\sum_{n=p}^{m}\sum_{j=p}^{m}\lambda_n\lambda_j\left|\int_{-\infty}^{\infty}\overline{\varphi_n(\xi)}\varphi_j(\xi)\,d\xi\right|^2=\sum_{n=p}^{m}\lambda_n^2,$$

which is valid for any positive integers $p<m$.

Besides, for any t the functional series on the variable ξ in (10) converges in the space $L^1(R)$. Indeed, this follows from the estimate

$$\int_{-\infty}^{\infty}\left|\sum_{n=p}^{m}\lambda_n\,\varphi_n(\xi+t)\overline{\varphi_n(\xi)}\right|d\xi\le\sum_{n=p}^{m}|\lambda_n|\int_{-\infty}^{\infty}\left|\varphi_n(\xi+t)\overline{\varphi_n(\xi)}\right|d\xi\le$$

$$\le\sum_{n=p}^{m}|\lambda_n|\left(\int_{-\infty}^{\infty}|\varphi_n(\xi+t)|^2\,d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}|\varphi_n(\xi)|^2\,d\xi\right)^{\frac{1}{2}}=\sum_{n=p}^{m}|\lambda_n|\int_{-\infty}^{\infty}|\varphi_n(\xi)|^2\,d\xi=\sum_{n=p}^{m}|\lambda_n|.$$

Define the function $K(\xi,\xi)$ by the equality

$$K(\xi,\xi)=\sum_{n=1}^{\infty}\lambda_n|\varphi_n(\xi)|^2, \tag{11}$$

where the functional series converges in the space $L^1(R)$. Evidently,

$$\int_a^b K(\xi,\xi)\,d\xi=\sum_{n=1}^{\infty}\lambda_n=\operatorname{sp}K. \tag{12}$$

Taking into account (10) and (11), we get

$$\int_{-\infty}^{\infty}|K(\xi+t,\xi)-K(\xi,\xi)|\,d\xi=\int_{-\infty}^{\infty}\left|\sum_{n=1}^{\infty}\lambda_n\,\overline{\varphi_n(\xi)}\{\varphi_n(\xi+t)-\varphi_n(\xi)\}\right|d\xi\le$$

$$\le \sum_{n=1}^{\infty} |\lambda_n| \int_{-\infty}^{\infty} |\overline{\varphi_n(\xi)}\{\varphi_n(\xi+t)-\varphi_n(\xi)\}| d\xi \le$$

$$\le \sum_{n=1}^{\infty} |\lambda_n| \left(\int_{-\infty}^{\infty} |\varphi_n(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |\varphi_n(\xi+t)-\varphi_n(\xi)|^2 d\xi\right)^{\frac{1}{2}} =$$

$$= \sum_{n=1}^{\infty} |\lambda_n| \left(\int_{-\infty}^{\infty} |\varphi_n(\xi+t)-\varphi_n(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

But (see [5], p. 499–502)

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |\varphi_n(\xi+t)-\varphi_n(\xi)|^2 d\xi = 0,$$

$$\left(\int_{-\infty}^{\infty} |\varphi_n(\xi+t)-\varphi_n(\xi)|^2 d\xi\right)^{\frac{1}{2}} \le \left(\int_{-\infty}^{\infty} |\varphi_n(\xi+t)|^2 d\xi\right)^{\frac{1}{2}} + \left(\int_{-\infty}^{\infty} |\varphi_n(\xi)|^2 d\xi\right)^{\frac{1}{2}} = 2.$$

Hence

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |K(\xi+t,\xi)-K(\xi,\xi)| d\xi = 0.$$

Thus, for any finite or infinite interval (α,β) the equality

$$\int_{\alpha}^{\beta} K(\xi,\xi)\, d\xi = \lim_{h\to 0} \frac{1}{h} \int_0^h \int_{\alpha}^{\beta} K(\xi+t,\xi)\, d\xi\, dt \tag{13}$$

holds. Particularly

$$\operatorname{sp} K = \lim_{h\to 0} \frac{1}{h} \int_0^h \int_a^b K(\xi+t,\xi)\, d\xi\, dt. \tag{14}$$

It is evident, that if the function $K(\xi,\eta)$ is continuous in the domain $(a,b)\times(a,b)$, then the function $K(\xi,\xi)$, defined in the usual sense, satisfies the equality (13) for any finite interval (α,β). Thus, for the function $K(\xi,\xi)$ the equality (12) is also true, as

$$\operatorname{sp} K = \lim_{h\to 0} \frac{1}{h} \int_0^h \int_{-\infty}^{\infty} K(\xi+t,\xi)\, d\xi\, dt = \lim_{\alpha\to-\infty} \lim_{\beta\to\infty} \left(\lim_{h\to 0} \frac{1}{h} \int_0^h \int_{\alpha}^{\beta} K(\xi+t,\xi)\, d\xi\, dt \right).$$

It is clear that above assertions remain true also for the case, where the set of eigenvalues of K is finite.

Remark. In the case of a finite segment $[a,b]$ and continuous in the square $[a,b]\times[a,b]$ function $K(\xi,\eta)$ equality (12) is proved also in [3], p. 144–152. The proof uses Steklov smoothing operator $S_h\ (h>0)$, defined by the formula

$$(S_h x)(t) = \frac{1}{2h} \int_{t-h}^{t+h} x(\xi)\, d\xi.$$

But in [3] the following assertion is used also: if a function $x(t)$ is continuous on $[a,b]$ and equal to zero outside of $[a,b]$, then $(S_h x)(t)$ converges to $x(t)$ uniformly on $[a,b]$ when $h\to 0$. This assertion, in general, is erroneous, since

$$\lim_{h\to 0} (S_h x)(a) = \frac{1}{2} x(a), \quad \lim_{h\to 0} (S_h x)(b) = \frac{1}{2} x(b).$$

Corollary. Let $H=L^2(a,b)$ with finite or infinite interval (a,b), and for some number $c>0$ the operators $A=V^*V-cI$ and $\mathrm{B}=VV^*-cI$ are integral operators, defined for $x\in L^2(a,b)$ by

$$(Ax)(\xi)=\int_a^b A(\xi,\eta)\,x(\eta)\,d\eta,\quad (Bx)(\xi)=\int_a^b B(\xi,\eta)\,x(\eta)\,d\eta,\quad \xi\in(a,b),$$

where the functions $A(\xi,\eta)$ and $B(\xi,\eta)$ satisfy the following conditions:

$$\int_a^b\int_a^b |A(\xi,\eta)|^2\,d\eta\,d\xi<\infty,\quad \int_a^b\int_a^b |B(\xi,\eta)|^2\,d\eta\,d\xi<\infty.$$

Then

$$\operatorname{ind}V=\frac{1}{c^2}\int_a^b\int_a^b\{|A(\xi,\eta)|^2-|B(\xi,\eta)|^2\}\,d\eta\,d\xi.$$

Besides, if operators A and B belong to the trace class, then

$$\operatorname{ind}V=\lim_{h\to 0}\frac{1}{hc}\int_0^h\int_a^b\{B(\xi+t,\xi)-A(\xi+t,\xi)\}\,d\xi\,dt$$

(we suppose that $A(\xi,\eta)$ and $B(\xi,\eta)$ are equal to zero outside of $(a,b)\times(a,b)$), and if the functions $A(\xi,\eta)$ and $B(\xi,\eta)$ are continuous in $(a,b)\times(a,b)$, then

$$\operatorname{ind}V=\frac{1}{c}\int_a^b\{B(\xi,\xi)-A(\xi,\xi)\}\,d\xi. \tag{15}$$

As an example of a bounded linear operator V in $L^2(0,\infty)$, for which $A=V^*V-I$ and $\mathrm{B}=VV^*-I$ are integral operators, we can take the operator, defined for $x\in L^2(0,\infty)$ by

$$(Vx)(\xi)=\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{m}\int_0^\infty x(\eta)\,S_k(\eta)\,e^{i\omega_k\xi\eta}\,d\eta,\quad \xi\in(0,\infty), \tag{16}$$

where $m\geq 1$ is an integer, i is the imaginary unit,

$$\omega_k=exp\left(\frac{i\pi k}{m}\right),\quad k=0,1,\cdots,m,$$

the functions $S_k(\eta)$ are continuous and bounded on $(0,\infty)$ with $S_m(\eta)\equiv 1$, $|S_0(\eta)|\equiv 1$, the function $S_0(\eta)$ has continuous and integrable on $(0,\infty)$ derivative $S'_0(\eta)$, and the limits $S_0(0)$ and $S_0(\infty)$ of $S_0(\eta)$ at $\eta\to 0$ and $\eta\to\infty$ are real numbers.

The adjoint operator V^* is defined by the formula

$$(V^*x)(\xi)=\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{m}\overline{S_k(\xi)}\int_0^\infty x(\eta)\,e^{-i\overline{\omega}_k\xi\eta}\,d\eta.$$

It is easy to see that

$$(Ax)(\xi)=\int_0^\infty A(\xi,\eta)\,x(\eta)\,d\eta,\quad (Bx)(\xi)=\int_0^\infty B(\xi,\eta)\,x(\eta)\,d\eta,$$

where

$$A(\xi,\eta)=\frac{1}{2\pi i}\sum_{k,j=0}^{m}\frac{\overline{S_k(\xi)}S_j(\eta)}{\overline{\omega}_k\xi-\omega_j\eta},$$

$$B(\xi,\eta)=\frac{1}{2\pi i(\xi+\eta)}\{\int_0^\infty\overline{S'_0(t)}e^{-it(\xi+\eta)}\,dt-\int_0^\infty S'_0(t)\,e^{it(\xi+\eta)}\,dt\}+$$

$$+\frac{1}{2\pi}\sum_{k=1}^{m-1}\int_0^\infty\{S_k(t)\,e^{it(\omega_k\xi+\eta)}+\overline{S_k(t)}\,e^{-it(\xi+\overline{\omega}_k\eta)}\}\,dt+$$

$$+\frac{1}{2\pi}\sum_{k=1}^{m-1}\int_0^\infty\{S_k(t)\overline{S_0(t)}e^{it(\omega_k\xi-\eta)}+\overline{S_k(t)}\,S_0(t)\,e^{it(\xi-\overline{\omega}_k\eta)}\}\,dt+$$

$$+\frac{1}{2\pi}\sum_{k,j=1}^{m-1}\int_0^\infty\overline{S_k(t)}\,S_j(t)\,e^{it(\omega_j\xi-\overline{\omega}_k\eta)}\,dt.$$

Under some additional restrictions on the functions S_k, the equality (15) can be proved and reduced to the form

$$\operatorname{ind}V=\frac{1}{2\pi i}\int_0^\infty\frac{S'_0(\xi)}{S_0\xi)}d\xi-\frac{1}{4}(S_0(\infty)-S_0(0)).$$

In the case of $m=1$ at least one of the operators V and V^* has inverse, even if the function S_0 is only measurable and bounded (see [6]).

Operator of the form (16) arises in the investigations of the scattering inverse problem for differential operator of order $2m$, and the equality (15) expresses a relation between scattering data (see [7–9]).

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Գծային սահմանափակ օպերատորի ինդեքսի համար որոշ բանաձևերի մասին

Անվերջ չափանի սեպարաբել հիլբերտյան տարածությունում գործող գծային սահմանափակ օպերատորի ինդեքսի համար արտածվում են բանաձևեր, որոնք կարող են օգտագործվել ինդեքսը հաշվելու համար:

О некоторых формулах для индекса линейного ограниченного оператора

Выводятся формулы для индекса действующего в бесконечномерном сепарабельном гильбертовом пространстве линейного ограниченного оператора, которые могут быть использованы для вычисления индекса.