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Mathematics

BOUNDARY VALUE PROBLEM FOR THE PSEUDOPARABOLIC EQUATIONS

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In the present paper the boundary value problem for the Sobolev type equation

$$\begin{cases} \frac{\partial}{\partial t} L(u(t,x)) + M(u(t,x)) = f(t,x), \quad t>0, \ x=(x_1,\dots,x_n)\in\Omega\subset\mathbb{R}^n, \\ u|_{\partial\Omega}=0, \\ (Lu)(0,x)=g(x), \quad x\in\Omega, \end{cases}$$

is considered, where L and M are second-order differential operators. It is proved that under some conditions this problem in the corresponding space has the unique solution.

Keywords**:** Sobolev type equations, pseudoparabolic equations, monotone and radial operators.

1. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with the smooth boundary Γ. We consider the following boundary value problem:

$$\begin{cases} \frac{\partial}{\partial t} L(u(t,x)) + M(u(t,x)) = f(t,x), \quad t>0, \ x=(x_1,\dots,x_n)\in\Omega\subset\mathbb{R}^n, & (1) \\ u|_{\partial\Omega}=0, & (2) \\ (Lu)(0,x)=g(x), \quad x\in\Omega, & (3) \end{cases}$$

where $L(u) = -\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\left(b_{ij}(t,x)\frac{\partial u}{\partial x_j}\right)$, $M(u) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$,

$$f(t,x)\in L_2\left((0,T);W_2^{-1}(\Omega)\right),\ g(x)\in W_2^{-1}(\Omega).$$

We suppose that the functions $b_{ij}(t,x)$ and $a_{ij}(t,x)$ $(i,j=1,2,\dots,n)$ are defined in $[0,T]\times\bar{\Omega}$, $b_{ij}(t,x)=b_{ji}(t,x)$, $a_{ij}(t,x)=a_{ji}(t,x)$ $(i,j=1,2,\dots,n)$ and

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for every $t \in [0,T]$ and $x \in \bar{\Omega}$ the following quadratic form is positively defined:

$$\sum_{i,j=1}^{n} b_{ij}(t,x)\xi_i\xi_j \ge c_0 |\xi|^2 , \tag{4}$$

where $\xi = (\xi_1,\dots,\xi_n)$, $c_0 = const > 0$.

The case of the problem (1)–(3) with $u|_{t=0} = g(x)$, instead of (3) (first boundary value problem), has been considered by R.A. Aleksandrian [1], G.S. Hakobyan, R.L. Shakhbaghyan [2], Kh. Gaevskii, K. Greger, K. Zakharis [3], R.E. Showalter [4], H.A.Mamikonyan [5] etc.

In this paper we study a new boundary value problem.

For the fixed $t \in [0,T]$ we define mappings $L(t)$ and $M(t)$ from $\overset{\circ}{W}{}_2^1(\Omega)$ to $W_2^{-1}(\Omega)$ by formulas

$$\langle L(t)v, w\rangle = \sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x)\frac{\partial v}{\partial x_i}\cdot\frac{\partial w}{\partial x_j}dx , \tag{5}$$

$$\langle M(t)v, w\rangle = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(t,x)\frac{\partial v}{\partial x_i}\cdot\frac{\partial w}{\partial x_j}dx , \tag{6}$$

where $v \in \overset{\circ}{W}{}_2^1(\Omega)$ and $w \in \overset{\circ}{W}{}_2^1(\Omega)$. It is easy to see that for $\forall v \in \overset{\circ}{W}{}_2^1(\Omega)$ formulas (5) and (6) define linear bounded functionals $L(t)v$ and $M(t)v$, which belong to $W_2^{-1}(\Omega)$. At the same time differential expressions $L(u)$ and $M(u)$ generate operators $(Lu)(t) = L(t)u(t,x)$ and $(Mu)(t) = M(t)u(t,x)$ that map $L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)$ into $L_2\left(0,T;W_2^{-1}(\Omega)\right)$.

Let's give some definitions (see [3, 4]). Let X be a real, reflexive Banach space.

Definition 1. The operator $A: X \to X^*$ is called

- radially continuous, if for $\forall x,y \in X$ the function $\varphi(s) = \langle A(x+sy), y\rangle$ is continuous in $[0,1]$;
- Lipschitz-continuous, if there exists a positive constant M such that
$$\|Ax - Ay\|_* \le M\|x-y\| \quad \text{for } \forall x,y \in X ;$$
- monotone, if $\langle Ax - Ay, x-y\rangle \ge 0$ for $\forall x,y \in X$;
- strictly monotone, if there exists a positive constant m such that
$$\langle Ax - Ay, x-y\rangle \ge m\|x-y\|^2 \quad \text{for } \forall x,y \in X .$$

Lemma 1. The operators $L(t)$, $M(t)$: $\overset{\circ}{W}{}_2^1(\Omega) \to W_2^{-1}(\Omega) = \left(\overset{\circ}{W}{}_2^1(\Omega)\right)^*$ are radially continuous and uniformly bounded with respect to t.

Proof. Indeed, for every functions $v(x), w(x) \in \overset{\circ}{W}{}_2^1(\Omega)$ we have

$$\varphi(s) = \langle L(t)(v+sw), w\rangle = \sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x)\frac{\partial(v+sw)}{\partial x_i}\cdot\frac{\partial w}{\partial x_j}dx =$$

$$= \sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x)\frac{\partial v}{\partial x_i}\cdot\frac{\partial w}{\partial x_j}dx + s\sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x)\frac{\partial w}{\partial x_i}\cdot\frac{\partial w}{\partial x_j}dx = \langle L(t)\, v,w\rangle + s\,\langle L(t)\, w,w\rangle.$$

Similarly, we get $\varphi(s) = \langle M(t)(v+sw), w\rangle = \langle M(t)v, w\rangle + s\langle M(t)w, w\rangle$, hence the functionals $\varphi(s)$ and $\psi(s)$ are linear. From the formulas (5) and (6) it follows that

$$\left|\langle L(t)v, w\rangle\right| = \left|\sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x)\frac{\partial v}{\partial x_i}\cdot\frac{\partial w}{\partial x_j}dx\right| \le$$

$$\le \sum_{i,j=1}^{n} \int_{\Omega} \left|b_{ij}(t,x)\right|\left|\frac{\partial v}{\partial x_i}\right|\left|\frac{\partial w}{\partial x_j}\right|dx \le c_1 \|v\|_{\overset{\circ}{W}{}_2^1(\Omega)} \cdot \|w\|_{\overset{\circ}{W}{}_2^1(\Omega)},$$

hence we have $\|L(t)v\|_* \le c_1\|v\|_{\overset{\circ}{W}{}_2^1}$, $\|M(t)v\|_* \le c_1\|v\|_{\overset{\circ}{W}{}_2^1}$. Now the Lipschitz continuity of the operators $L(t)$ and $M(t)$ follows from their linearity.

Lemma 2. The operators $L(t)$ are uniformly strictly monotone with respect to t.

Proof. From condition (4) it follows that for every $v(x), w(x) \in \overset{\circ}{W}{}_2^1(\Omega)$ we have

$$\left|\langle L(t)v - L(t)w, v-w\rangle\right| = \langle L(t)(v-w), v-w\rangle = \sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x)\frac{\partial(v-w)}{\partial x_i}\cdot\frac{\partial(v-w)}{\partial x_j}dx \ge$$

$$\ge c_0 \int_{\Omega} \sum_{i,j=1}^{n} \left|\frac{\partial(v-w)}{\partial x_i}\right|^2 dx = c_0\|v-w\|^2_{\overset{\circ}{W}{}_2^1(\Omega)}.$$

Definition 2. Let X and Y be linear spaces and $s=[0,T]$. A mapping $G: L_2(0,T;X) \to L_2(0,T;Y)$ is called Volterra-type, if from the condition $u(s) = v(s)$ for almost all $s \in [0,t]$, $t \in S$, it follows that $(Gu)(s) = (Gv)(s)$ for almost all $s \in [0,t]$. It is evident that the operator M is of Volterra-type.

From the Lemma 1, Lemma 2 and Lemma 2.2 (see [1]) we get

Theorem 1. The operator $L: L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right) \to L_2\left(0,T;W_2^{-1}(\Omega)\right)$ is radially continuous, strictly monotone, and there exists the inverse operator L^{-1}, which is Lipschitz continuous and

$$\left(L^{-1}f\right)(t) = L^{-1}(t)f(t) \quad \forall t \in [0,T], \quad \forall f \in L_2\left(0,T;W_2^{-1}(\Omega)\right).$$

Together with the problem (1)–(3), let's consider the following one

$$\begin{cases} v' + \mathbb{A}v = f, \\ v(0) = g, \end{cases} \tag{7}$$

where $\mathbb{A} = ML^{-1} : L_2\left(0,T;W_2^{-1}(\Omega)\right) \to L_2\left(0,T;W_2^{-1}(\Omega)\right)$. Since the operator $\mathbb{A}$ satisfies the conditions of Theorem 1.3 (see [3]), we conclude that the problem (7) has the unique solution. Denote it by v_*. Then the function $v_* = L^{-1}v$ is the solution of the problem (1)–(3). Thus, we can formulate the following (see Theorem 2.4, [3])

Theorem 2. Let the functions $b_{ij}(t,x) = b_{ji}(t,x)$, $a_{ij}(t,x) = a_{ji}(t,x)$ ($i,j = 1,2,\dots,n$) be continuous in the domain $[0,T]\times\Omega$, and condition (4) holds for any $t\in[0,T]$ and any $x\in\bar{\Omega}$. Then the problem (1)–(3) has a unique solution and $L(u)\in C\left(0,T;W_2^{-1}(\Omega)\right)$, $\frac{\partial}{\partial t}\left(L(u)\right)\in L_2\left(0,T;W_2^{-1}(\Omega)\right)$.

2. Now we consider the problem (1)–(3) with the assumption that the operators L and M are second order nonlinear differential operators:

$$L(u) = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(b_i(t,x,\nabla u)\right), \quad M(u) = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(a_i(t,x,\nabla u)\right),$$

where the functions $b_i(t,x,\xi_1,\dots,\xi_n)$, $a_i(t,x,\xi_1,\dots,\xi_n)$ are defined and continuous in $[0,T]\times\bar{\Omega}\times R^n$, and have continuous derivatives with respect to ξ_j ($j = 1,2,\dots,n$).

We suppose that the functions $b_i(t,x,\xi)$ and $a_i(t,x,\xi)$ ($\xi = (\xi_1,\dots,\xi_n)$, $i = 1,2,\dots,n$) satisfy the conditions:

1) $\left|b_i(t,x,\xi)\right| \le c_1\left(|\xi|+1\right)$, $c_1 = const > 0$, $i = 1,2,\dots,n$,

2) $\left|b_{ij}(t,x,\xi)\right| = \left|\frac{\partial b_1}{\partial \xi_j}\right| \le c_2$, $c_2 = const > 0$, $i,j = 1,2,\dots,n$,

3) $\sum_{i,j=1}^{n} b_{ij}(t,x,\xi)\eta_1\eta_2 \ge c_3|\eta|^2$ $\forall t\in[0,T]$, $\forall x\in\bar{\Omega}$ and $\forall \eta = (\eta_1,\dots,\eta_n)\in R^n$,

4) $\left|a_i(t,x,\xi)\right| \le c_4\left(|\xi|+1\right)$, $\left|\frac{\partial a_i(t,x,\xi)}{\partial \xi_j}\right| \le c_5$, $i,j = 1,2,\dots,n$.

For fixed $t\in[0,T]$ define the operators $L(t)$ and $M(t)$ from $\overset{\circ}{W}{}_2^1(\Omega)$ to $W_2^{-1}(\Omega)$ by formulas

$$\left\langle L(t)v,w\right\rangle = \sum_{i=1}^{n}\int_{\Omega} b_i(t,x,\nabla u)\frac{\partial w}{\partial x_i}dx, \tag{7}$$

$$\left\langle M(t)v,w\right\rangle = \sum_{i=1}^{n}\int_{\Omega} a_i(t,x,\nabla u)\frac{\partial w}{\partial x_i}dx. \tag{8}$$

Operators $L(t)$ and $M(t)$ $(t\in[0,T])$ generate mappings L and M from $L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)$ to $L_2\left(0,T;W_2^{-1}(\Omega)\right)$ by formulas

$$(Lu)(t)=L(t)(u(t,x)), \tag{9}$$

$$(Mu)(t)=M(t)(u(t,x)). \tag{10}$$

Lemma 3 ([5]). Let the conditions 1)–3) hold. Then the operator $L(t)$ is radially continuous and strictly monotone.

Proof. For every $s_1,s_2\in[0,1]$ we have

$$|\varphi(s_1)-\varphi(s_2)|=\left|\langle L(t)(u_1+s_1v),v\rangle-\langle L(t)(u_2+s_2v),v\rangle\right|=$$

$$=\left|\langle L(t)(u+s_1v)-L(t)(u+s_2v),v\rangle\right|=$$

$$=\left|\sum_{i=1}^{n}\int_{\Omega}b_i(t,x,\nabla u+s_1\nabla v)-b_i(t,x,\nabla u+s_2\nabla v)\frac{\partial v}{\partial x_i}dx\right|=$$

$$=\left|\sum_{i=1}^{n}\int_{\Omega}\int_0^1\left(\sum_{j=1}^{n}\frac{\partial b_i(t,x,\nabla u+s_1\nabla v+\tau(s_2-s_1)\nabla v)}{\partial\xi_j}(s_2-s_1)\frac{\partial v}{\partial x_i}\cdot\frac{\partial v}{\partial x_j}\right)dtdx\right|\le C|s_1-s_2|\,\|v\|^2_{\overset{\circ}{W}{}_2^1(\Omega)},$$

thus, the operator $L(t)$ is radially continuous.

Now we prove that the operator $L(t)$ is strictly monotone. Indeed, from the condition 3) we get

$$\sum_{i=1}^{n}\int_{\Omega}\left[b_i(t,x,\nabla u)-b_i(t,x,\nabla v)\right]\frac{\partial(u-v)}{\partial x_i}dx=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\Omega}\int_0^1 b_{ij}(t,x,\nabla v+\tau(\nabla u-\nabla v))\frac{\partial(u-v)}{\partial x_i}\cdot\frac{\partial(u-v)}{\partial x_j}dtdx\ge c_3\|u-v\|^2_{\overset{\circ}{W}{}_2^1(\Omega)}.$$

The proof of Lemma 3 is complete.

It is easy to verify that the operator

$$M:L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)\to L_2\left(0,T;W_2^{-1}(\Omega)\right)$$

is Lipschitz continuous and of Voltera type. From Lemma 2 and Lemma 2.2 (see [3]) it immediately follows

Lemma 4. The operator $L:L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)\to L_2\left(0,T;W_2^{-1}(\Omega)\right)$ is radially continuous, strictly monotone, and there exists the inverse operator

$$L^{-1}:L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)\to L_2\left(0,T;W_2^{-1}(\Omega)\right),$$

whereas $(L^{-1}f)(t)=L^{-1}(t)f(t)$ for $\forall t\in[0,T]$ and $\forall f\in L_2\left(0,T;W_2^{-1}(\Omega)\right)$.

From Lemma 4 and Theorem 2.4 (see [3]) it immediately follows

Theorem 3. Let the functions $b_i(t,x,\xi)$ and $a_i(t,x,\xi)$ ($i=1,2,\ldots,n$) satisfy the conditions 1)–4). Then the problem (1)–(3), where the operators L and M are defined by formulas (9) and (10), has a unique solution, and $L(u)\in C\left(0,T;W_2^{-1}(\Omega)\right)$, $\frac{\partial}{\partial t}L(u)\in L_2\left(0,T;W_2^{-1}(\Omega)\right)$.

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Սիավաշ Ղորբանիան

Եզրային խնդիր պսևդոպարաբոլական հավասարումների համար

Աշխատանքում ուսումնասիրվում է Սոբոլևի հավասարումների տիպի հավասարումների մի դասի համար հետևյալ սկզբնական-եզրային խնդիրը.

$$\begin{cases} \frac{\partial}{\partial t} L(u(t,x)) + M(u(t,x)) = f(t,x), \quad t>0, \ x=(x_1,\ldots,x_n)\in\Omega\subset\mathbb{R}^n, \\ u|_{\partial\Omega}=0, \\ (Lu)(0,x)=g(x), \qquad x\in\Omega, \end{cases}$$

որտեղ L-ը և M-ը 2-րդ կարգի դիֆերենցիալ օպերատորներ են: Ապա-ցուցվում է, որ որոշակի պայմանների դեպքում համապատասխան ֆունցիոնալ տարածությունում այդ խնդիրն ունի լուծում և այն է միակը:

Сиаваш Гарбаниан

Краевая задача для псевдопараболических уравнений

В работе исследуется начально-краевая задача для уравнения типа Соболева

$$\begin{cases} \frac{\partial}{\partial t} L(u(t,x)) + M(u(t,x)) = f(t,x), \quad t>0, \ x=(x_1,\ldots,x_n)\in\Omega\subset\mathbb{R}^n, \\ u|_{\partial\Omega}=0, \\ (Lu)(0,x)=g(x), \qquad x\in\Omega, \end{cases}$$

где L и M – дифференциальные операторы второго порядка. Доказывается, что если удовлетворяются некоторые условия, то эта задача в соответствующем функциональном пространстве имеет единственное решение