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Mathematics

ON AN ANISOTROPIC BOUNDARY PROBLEM OF DIFFRACTION WITH FIRST AND SECOND TYPE BOUNDARY CONDITIONS

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In the present paper solvability of a class of boundary problems associated with the anisotropic Helmholtz–Shrodinger equation in the upper and lower semiplanes of Sobolev spaces is studied. The first and second type boundary conditions are assumed to hold on the line $y=0$. Solvability of these boundary problems reduces to solvability of Riman–Hilbert boundary problem. The solvability analysis is based on the factorization problem of some matrix-function.

***Keywords*:** Helmholtz–Shrodinger equation, factorization of matrix-functons.

Let us formulate the following boundary problem. Denote $\Omega^{\pm}=\{(x,y)\in R^2 : y \gtrless 0\}$. Let $H^{1/2}(\Omega^{\pm})$, $H^{-1/2}(\Omega^{\pm})$ be Sobolev spaces [1]. Consider the following anisotropic system of the Helmholtz–Shrodinger equations:

$$\begin{cases} \Delta u+\left(k_+^2+2\beta_+^2\operatorname{sech}^2(\beta_+ y)\right)u=0 \ \text{ in } \Omega^+, \\ \Delta u+\left(k_-^2+2\beta_-^2\operatorname{sech}^2(\beta_- y)\right)u=0 \ \text{ in } \Omega^-, \end{cases} \tag{1}$$

with boundary conditions:

$$\begin{cases} \begin{cases} a_0u(x,+0)+b_0u(x,-0)=h_0(x) \\ a_1\dfrac{\partial u(x,+0)}{\partial y}+b_1\dfrac{\partial u(x,-0)}{\partial y}=h_1(x) \end{cases} \text{ in } R^+, \\ \begin{cases} c_0u(x,+0)+d_0u(x,-0)=p_0(x) \\ c_1\dfrac{\partial u(x,+0)}{\partial y}+d_1\dfrac{\partial u(x,-0)}{\partial y}=p_1(x) \end{cases} \text{ in } R^-, \end{cases} \tag{2}$$

where $\operatorname{Im}(k_\pm)>0$, coefficients $a_0,a_1,b_0,b_1,c_0,c_1,d_0,d_1$ are complex constants. Note that $h_0\in H^{1/2}(R^+)$, $h_1\in H^{-1/2}(R^+)$, $p_0\in H^{1/2}(R^-)$, $p_1\in H^{-1/2}(R^-)$, and one should search for the solution of the boundary problem (1), (2) in the space $L^2(R^2)$. Problems of this type were first studied by A.J. Sommerfeld for the wave diffraction on the interface of two media [2–6]. Applying Fourier integral

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transform to the solution $u \in L^2(R^2)$ over the variable x, one derives the following system of ordinary differential equations under the assumption $\operatorname{Im}(\varkappa_{\pm}(\lambda)) > 0$:

$$\begin{cases} \dfrac{d^2 \hat{u}}{dy^2} + (\varkappa_+^2(\lambda) + 2\beta_+^2 \operatorname{sech}^2(\beta_+ y))\, \hat{u} = 0 \quad \text{for } y > 0, \\ \dfrac{d^2 \hat{u}}{dy^2} + (\varkappa_-^2(\lambda) + 2\beta_-^2 \operatorname{sech}^2(\beta_- y))\, \hat{u} = 0 \quad \text{for } y < 0. \end{cases} \tag{3}$$

Then $\hat{u} \in L^2(R^2)$. Denote $\gamma_{\pm}(\lambda) = \sqrt{\lambda^2 - k_{\pm}^2} = i\varkappa_{\pm}(\lambda) = \sqrt{k_{\pm}^2 - \lambda^2}$. Thus, the general solution of the system of ordinary differential equations in the $L^2(R^2)$-space has the following form:

$$\hat{u}(\lambda, y) = \begin{cases} a(\lambda) \dfrac{i\varkappa_+(\lambda) - \beta_+ \tanh(\beta_+ y)}{i\varkappa_+(\lambda)} e^{i\varkappa_+(\lambda) y} \quad \text{for } y > 0, \\ b(\lambda) \dfrac{i\varkappa_-(\lambda) + \beta_- \tanh(\beta_- y)}{i\varkappa_-(\lambda)} e^{-i\varkappa_-(\lambda) y} \quad \text{for } y < 0. \end{cases} \tag{4}$$

Let $\chi_{\pm}(y) = 1/2(1 \pm \operatorname{sgn} y)$ and

$$\begin{cases} \hat{u}_+(\lambda, y) = \dfrac{\chi_+(y)}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} u(x, y) e^{i\lambda x} dx, \\ \hat{u}_-(\lambda, y) = \dfrac{\chi_-(y)}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} u(x, y) e^{i\lambda x} dx. \end{cases} \tag{5}$$

Then from equation (4) it follows that

$$\hat{u}(\lambda, y) = \hat{u}_+(\lambda, y) + \hat{u}_-(\lambda, y). \tag{6}$$

We introduce the following notations:

$$\begin{cases} u_-(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{0} (a_0 u(x, +0) + b_0 u(x, -0) - h_0(x)) e^{i\lambda x} dx, \\ w_-(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{0} (a_1 \dfrac{\partial u(x, +0)}{y} + b_1 \dfrac{\partial u(x, -0)}{y} - h_1(x)) e^{i\lambda x} dx. \end{cases} \tag{7}$$

Similarly

$$\begin{cases} u_+(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{0}^{\infty} (c_0 u(x, +0) + d_0 u(x, -0) - p_0(x)) e^{i\lambda x} dx, \\ w_+(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{0}^{\infty} (c_1 \dfrac{\partial u(x, +0)}{y} + d_1 \dfrac{\partial u(x, -0)}{y} - p_1(x)) e^{i\lambda x} dx. \end{cases} \tag{8}$$

It is easy to see that

$$\frac{d\hat{u}(\lambda, y)}{dy} = \begin{cases} a(\lambda) \left[(i\varkappa_+(\lambda) - \beta_+ \tanh(\beta_+ y)) - \dfrac{\beta_+^2}{i\varkappa_+(\lambda) \cosh^2(\beta_+ y)} \right] e^{i\varkappa_+(\lambda) y} \quad \text{for } y > 0, \\ -b(\lambda) \left[(i\varkappa_-(\lambda) + \beta_- \tanh(\beta_- y)) + \dfrac{\beta_-^2}{i\varkappa_-(\lambda) \cosh^2(\beta_- y)} \right] e^{-i\varkappa_-(\lambda) y} \quad \text{for } y < 0. \end{cases} \tag{9}$$

Using boundary conditions (2) and taking into account equations (4) and (9), one derives

$$\begin{cases} a_0 a(\lambda) + b_0 b(\lambda) = u_-(\lambda) + \hat{h}_0(\lambda), \\ \dfrac{-a_1[\varkappa_+^2(\lambda)+\beta_+^2]a(\lambda)}{i\varkappa_+(\lambda)} + \dfrac{b_1[\varkappa_-^2(\lambda)+\beta_-^2]b(\lambda)}{i\varkappa_-(\lambda)} = w_-(\lambda) + \hat{h}_1(\lambda), \end{cases} \tag{10}$$

where $\hat{h}_0(\lambda) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} h_0(x) e^{i\lambda x} dx,\ \hat{h}_1(\lambda) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} h_1(x) e^{i\lambda x} dx$. Hereof we assume that the determinant $\Delta(\lambda)$ of system (10) is not zero, i.e.

$$\Delta(\lambda) = a_0 b_1 \frac{\varkappa_-^2(\lambda)+\beta_-^2}{i\varkappa_-(\lambda)} + a_1 b_0 \frac{\varkappa_+^2(\lambda)+\beta_+^2}{i\varkappa_+(\lambda)} = a_0 b_1 \frac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)} + a_1 b_0 \frac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)} \neq 0. \tag{11}$$

Since

$$\begin{cases} a(\lambda) = \dfrac{1}{\Delta(\lambda)} \left\{ b_1 \dfrac{\varkappa_-^2(\lambda)+\beta_-^2}{i\varkappa_-(\lambda)} (u_-(\lambda) + \hat{h}_0(\lambda)) - b_0 (w_-(\lambda) + \hat{h}_1(\lambda)) \right\}, \\ b(\lambda) = \dfrac{1}{\Delta(\lambda)} \left\{ a_1 \dfrac{\varkappa_+^2(\lambda)+\beta_+^2}{i\varkappa_+(\lambda)} (u_-(\lambda) + \hat{h}_0(\lambda)) + a_0 (w_-(\lambda) + \hat{h}_1(\lambda)) \right\}, \end{cases} \tag{12}$$

then, taking into account that

$$\begin{cases} u_+(\lambda) = c_0 a(\lambda) + d_0 b(\lambda) - \hat{p}_0(\lambda), \\ w_+(\lambda) = \dfrac{-c_1[\varkappa_+^2(\lambda)+\beta_+^2]a(\lambda)}{i\varkappa_+(\lambda)} + \dfrac{d_1[\varkappa_-^2(\lambda)+\beta_-^2]b(\lambda)}{i\varkappa_-(\lambda)} - \hat{p}_1(\lambda), \end{cases} \tag{13}$$

where $\hat{p}_0(\lambda) = \dfrac{1}{\sqrt{2\pi}} \int_0^{\infty} p_0(x) e^{i\lambda x} dx,\ \hat{p}_1(\lambda) = \dfrac{1}{\sqrt{2\pi}} \int_0^{\infty} p_1(x) e^{i\lambda x} dx$, one derives the following Riman–Hilbert boundary problem with respect to vector-functions

$$\vec{u}_+(\lambda) = \begin{Bmatrix} u_+(\lambda) \\ w_+(\lambda) \end{Bmatrix}, \quad \vec{u}_-(\lambda) = \begin{Bmatrix} u_-(\lambda) \\ w_-(\lambda) \end{Bmatrix}, \tag{14}$$

that are analytical functions in the upper and lower semiplanes respectively.

In vector notations this problem takes the following form:

$$\vec{u}_+(\lambda) = L(\lambda)\vec{u}_-(\lambda) + \vec{m}(\lambda), \tag{15}$$

where the matrix function $L(\lambda)$ may be written as

$$L(\lambda) = \frac{1}{\Delta(\lambda)} \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) \end{pmatrix} \tag{16}$$

with

$$A_{11}(\lambda) = a_1 d_0 \frac{\varkappa_+^2(\lambda)+\beta_+^2}{i\varkappa_+(\lambda)} + b_1 c_0 \frac{\varkappa_-^2(\lambda)+\beta_-^2}{i\varkappa_-(\lambda)} = a_1 d_0 \frac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)} + b_1 c_0 \frac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)},$$

$$A_{12}(\lambda) = a_0 d_0 - b_0 c_0,$$

$$A_{21}(\lambda) = (a_1 d_1 - b_1 c_1) \frac{\varkappa_+^2(\lambda)+\beta_+^2}{i\varkappa_+(\lambda)} \cdot \frac{\varkappa_-^2(\lambda)+\beta_-^2}{i\varkappa_-(\lambda)} = (a_1 d_1 - b_1 c_1) \frac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)} \cdot \frac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)},$$

$$A_{22}(\lambda) = b_0 c_1 \frac{\varkappa_+^2(\lambda)+\beta_+^2}{i\varkappa_+(\lambda)} + a_0 d_1 \frac{\varkappa_-^2(\lambda)+\beta_-^2}{i\varkappa_-(\lambda)} = b_0 c_1 \frac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)} + a_0 d_1 \frac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)}.$$

The coordinates of the vector-function $\vec{m}(\lambda)=\begin{Bmatrix} m_1(\lambda) \\ m_2(\lambda) \end{Bmatrix}$ have the following form:

$$m_1(\lambda)=\frac{\hat{h}_0(\lambda)}{\Delta(\lambda)}\left\{a_1 d_0 \frac{\varkappa_+^2(\lambda)+\beta_+^2}{i\varkappa_+(\lambda)}+b_1 c_0 \frac{\varkappa_-^2(\lambda)+\beta_-^2}{i\varkappa_-(\lambda)}\right\}+\frac{a_0 d_0-b_0 c_0}{\Delta(\lambda)}\hat{h}_1(\lambda)-\hat{p}_0(\lambda),$$

$$m_2(\lambda)=\frac{\hat{h}_1(\lambda)}{\Delta(\lambda)}\left\{b_0 c_1 \frac{\varkappa_+^2(\lambda)+\beta_+^2}{i\varkappa_+(\lambda)}+a_0 d_1 \frac{\varkappa_-^2(\lambda)+\beta_-^2}{i\varkappa_-(\lambda)}\right\}+$$

$$+\frac{a_1 d_1-b_1 c_1}{\Delta(\lambda)}\hat{h}_0(\lambda)\frac{\varkappa_+^2(\lambda)+\beta_+^2}{i\varkappa_+(\lambda)}\cdot\frac{\varkappa_-^2(\lambda)+\beta_-^2}{i\varkappa_-(\lambda)}-\hat{p}_1(\lambda).$$

The case of $\beta_-=\beta_+=0$ was studied in the papers [2, 6]. So there holds the following

Theorem. If the function $u\in L^2(R^2)$ is the solution of the boundary problem (1), (2), then the pair of vector-functions $\vec{u}_+(\lambda)$ and $\vec{u}_-(\lambda)$ is the solution of the Riman–Hilbert boundary problem (15). And vice-versa, if one applies the inverse Fourier transform to the function $\hat{u}(\lambda,y)=\hat{u}_+(\lambda,y)+\hat{u}_-(\lambda,y)$, which is associated with vector functions $\vec{u}_+(\lambda)$ and $\vec{u}_-(\lambda)$ by relations (7), (8), one obtains the solution of the boundary problem (1), (2).

Theory of the Riman–Hilbert problem solvability is presented in monographs [7–10]. The essence of this problem's solution is related to factorization of matrix-function $L(\lambda)$, i.e. to its representation as $L(\lambda)=L_+(\lambda)\Lambda(\lambda)L_-(\lambda)$, where the matrix-functions $L_+(\lambda)$, $L_+^{-1}(\lambda)$ are analytic in the upper semiplane, and $L_-(\lambda)$, $L_-^{-1}(\lambda)$ are analytic in the lower semiplane. The diagonal matrix-function can be presented as

$$\Lambda(\lambda)=\begin{pmatrix} \left(\frac{\lambda-i}{\lambda+i}\right)^{\alpha_1} & 0 \\ 0 & \left(\frac{\lambda-i}{\lambda+i}\right)^{\alpha_2} \end{pmatrix}, \qquad (17)$$

where α_1 and α_2 are partial indices of the given diagonal matrix-function, which allow to write down the Fredholm characteristics of the initial boundary problem.

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Ս. Ա. Հուսեյն Մաթեկոլաեյի, Ա.Հ. Քամալյան, Մ.Ի. Կարախանյան

Առաջին և երկրորդ սեռի եզրային պայմաններով դիֆրակցիայի մի անիզոտրոպ եզրային խնդրի մասին

Աշխատանքում ուսումնասիրվում է Հելմհոլց–Շրեդինգերի անիզոտրոպ հավասարման եզրային խնդրի լուծելիությունը Սոբոլևի տարածություններում վերին և ներքին կիսահարթություններում: Եզրային խնդրի լուծելիության հարցը հանգեցվում է Ռիման–Հիլբերտի համապատասխան խնդրի լուծելիությանը:

С. А. Усейни Матеколаэи, А. Г. Камалян, М. И. Караханян

Об одной анизотропной граничной задаче дифракции с граничными условиями первого и второго рода

В работе изучается разрешимость анизотропной задачи уравнения Гельмголца–Шредингера в пространствах Соболева в верхней и нижней полуплоскостях. Разрешимость этой граничной задачи сводится к разрешимости соответствующей задачи Римана–Гильберта.