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Mathematics

A MIXED PROBLEM FOR THE FOURTH ORDER DEGENERATE ORDINARY DIFFERENTIAL EQUATION

ESMAIL YOUSEFI*

Azad University of Karraj, Iran

A mixed problem for the equation

$$Lu \equiv (t^{\alpha} u'')'' + au = f, \qquad (1)$$

where $0 \le \alpha \le 4$, $t \in [0,b]$, $f \in L_2(0,b)$, is considered. Firstly, the weighted Sobolev spaces W_{α}^2, $W_{\alpha}^2(0)$, $W_{\alpha}^2(b)$ and the generalized solution to equation (1) are defined. Next, the existence and uniqueness of the generalized solution for the mixed problem is studied, as well as the description of the spectrum of corresponding operator is given.

***Keywords*:** mixed problem, weighted Sobolev spaces, generalized solution, spectrum of linear operators.

1. Problem Formulation. Consider the mixed problem for the following ordinary differential equation of the fourth order

$$Lu \equiv (t^{\alpha} u'')'' + au = f, \qquad (1)$$

where $0 \le \alpha \le 4$, $t \in [0,b]$, $f \in L_2(0,b)$, $a = const$.

Define the weighted Sobolev spaces W_{α}^2, $W_{\alpha}^2(0)$, $W_{\alpha}^2(b)$ and consider the behavior of functions from these spaces in neighborhood of $t = 0$. Then, define the generalized solution to the mixed problem for the equation (1). The existence and uniqueness of the generalized solution are proved under some conditions on coefficient a. Moreover, we have to give a description of the spectrum of operator L and the operator domain $D(L)$.

Note that the Dirichlet problem for degenerate ordinary differential equations of second and fourth orders have been considered in [1, 2], and for higher orders – in [3].

2. The Spaces W_{α}^2, $W_{\alpha}^2(0)$, $W_{\alpha}^2(b)$. Let $\alpha \ge 0$, and t belongs to the finite interval $(0,b)$. Consider the set W_{α}^2 of the functions $u(t)$, which have generalized derivatives of second order, such that the semi-norm $\|u\|_1 = \int_0^b t^{\alpha} |u''(t)|^2 \, dt$ is finite.

* E-mail: u3phi@kiau.ac.ir

First note, that for the functions $u \in W_\alpha^2$ for every $t_0 \in (0,b]$ there exist finite values $u(t_0)$ and $u'(t_0)$ (see [3]). Below we study the behavior of functions $u(t)$ and $u'(t)$ in neighborhood of $t = 0$.

Proposition 1. For $u \in W_\alpha^2$ the following inequalities hold:

$$|u(t)|^2 \le \left(c_1 + c_2 t^{3-\alpha}\right)\|u\|_1^2, \ \alpha \ne 1, \ \alpha \ne 3, \tag{2}$$

$$|u'(t)|^2 \le \left(c_1 + c_2 t^{1-\alpha}\right)\|u\|_1^2, \ \alpha \ne 1. \tag{3}$$

In (2) $t^{3-\alpha}$ is replaced with $|\ln t|$ for $\alpha = 3$, $t^{3-\alpha}$ with $t^2|\ln t|$ for $\alpha = 1$; in (3) $t^{1-\alpha}$ is replaced with $|\ln t|$ (see [4]).

From the inequality (2) for $0 \le \alpha < 4$ we get the inequality

$$\|u\|_{L_2(0,b)} \le c\|u\|_1, \tag{4}$$

i.e. we have the following embedding

$$W_\alpha^2 \subset L_2(0,b). \tag{5}$$

The embedding (5) is true also for $\alpha = 4$ and fails for $\alpha > 4$ (see [4]).

Now we can define the following norm in W_α^2

$$\|u\|_{W_\alpha^2}^2 = \int_0^b \left(t^\alpha |u''(t)|^2 + |u(t)|^2\right)dt. \tag{6}$$

The space W_α^2 is a Hilbert space with scalar product $(u,v)_\alpha = \left(t^\alpha u'', v''\right) + (u,v)$, where $(\cdot,\cdot)$ stands for the scalar product in $L_2(0,b)$. Obviously, for $0 \le \alpha \le 4$ we have the following inequality

$$\|u\|_{L_2(0,b)} \le c\|u\|_{W_\alpha^2}. \tag{7}$$

Proposition 2. The embedding (5) for $0 \le \alpha < 4$ is compact.

Note also that for $\alpha = 4$ the continuous embedding (5) is not compact (see [2]).

Denote by $W_\alpha^2(0)$ the subspace of W_α^2, for which $u(0) = u'(0) = 0$, if they exist. Note that for $3 \le \alpha \le 4$ we have $W_\alpha^2(0) = W_\alpha^2$. As norm in the space $W_\alpha^2(0)$ we use the norm (6). For $0 \le \alpha < 1$ the co-dimension of the space $W_\alpha^2(0)$ in W_α^2 is equal to $\dim W_\alpha^2 / W_\alpha^2(0) = 2$ and for $1 \le \alpha < 3$ $\operatorname{co}\dim W_\alpha^2(0) = \dim W_\alpha^2 / W_\alpha^2(0) = 1$ (see Proposition 1).

Define by $W_\alpha^2(b)$ the subspace of W_α^2, for which $u(b) = u'(b) = 0$. As an equivalent norm to norm (6) we use $\|u\|_{W_\alpha^2(b)}^2 = \int_0^b t^\alpha |u''(t)|^2 dt$ in $W_\alpha^2(b)$. For $0 \le \alpha \le 4$ the co-dimension of the space $W_\alpha^2(b)$ in W_α^2 is equal to $\operatorname{co}\dim W_\alpha^2(b) = \dim W_\alpha^2 / W_\alpha^2(b) = 2$. Note that the embedding $W_\alpha^2(0) \subset L_2(0,b)$ and $W_\alpha^2(b) \subset L_2(0,b)$ are compact for $0 \le \alpha < 4$ (see Proposition 2).

3. The Mixed Problem. In this section we define the generalized solutions of the mixed problem for equation (1) in spaces $W_\alpha^2(0)$ and $W_\alpha^2(b)$.

Definition 1. The function $u \in W_\alpha^2(0)$ is called the generalized solution of the mixed problem for equation (1), if for every $v \in W_\alpha^2(0)$ the following equality holds:

$$\left(t^\alpha u'', v''\right) + a(u,v) = (f,v). \tag{8}$$

Note that, if the generalized solution $u \in W_\alpha^2(0)$ is classical, then for $\alpha = 0$ we get the following conditions (see [5]) $u(0) = u'(0) = u''(b) = u'''(b) = 0.$

Consider the particular case of equation (1) when $a = 1$:

$$Bu \equiv \left(t^\alpha u''\right)'' + u = f. \tag{9}$$

Proposition 3. For every $f \in L_2(0,b)$ the generalized solution of the mixed problem for equation (9) exists and is unique.

Proof. Uniqueness of the generalized solution for equation (9) immediately follows from the equality (8) (with $a = 1$), if we put $f = 0$ and $v = u$. To prove the existence we define the functional $l_f(v) = (f,v), f \in L_2(0,b)$ over the space $W_\alpha^2(0)$. Using the inequality (7) we get

$$\left|l_f(v)\right|^2 = \left|\int_0^b f(t)\overline{v(t)}dt\right|^2 \le \|f\|^2_{L_2(0,b)} \|v\|^2_{L_2(0,b)} \le c\|f\|^2_{L_2(0,b)} \|v\|_{W_\alpha^2},$$

i.e. $l_f(v)$ is a linear continuous functional over the space $W_\alpha^2(0)$. Using Riesz lemma on the functional representation we get $l_f(v) = (u_0, v)_\alpha$, $u_0 \in W_\alpha^2(0)$. Therefore, the function u_0 is the generalized solution for equation (9) (see [1]).

Define the operator $B : L_2(0,b) \to L_2(0,b)$ corresponding to Definition 1.

Definition 2. We say that the function $u \in W_\alpha^2(0)$ belongs to the domain $D(B)$ of operator B, if there exists $f \in L_2(0,b)$, such that the equality (8) is valid. In this case we write $Bu = f$.

Theorem 1. Operator $B : L_2(0,b) \to L_2(0,b)$ is positive and self-adjoint. The bounded operator $B^{-1} : L_2(0,b) \to L_2(0,b)$ for $0 \le \alpha < 4$ is compact.

Proof. The symmetry and positivity of the operator B is a direct consequence of Definition 2. The coincidence of $D(B)$ and $D(B^*)$ (B^* is the adjoint to operator B) follows from the existence of a generalized solution of (9) for every $f \in L_2(0,b)$ (see Proposition 3). Note that Definition 2 implies the inequality $\|u\|_{W_\alpha^2} \le c\|Bu\|_{L_2(0,b)}$.

The compactness of the operator B^{-1} for $0 \le \alpha < 4$ now follows from Proposition 2.

Corollary 1. For $0 \le \alpha < 4$ the operator B has a discrete spectrum, and its eigenfunction system is complete in $f \in L_2(0,b)$ (see [5]).

Note that now we can rewrite equation (1) in the form $Bu = (1-a)u + f$, i.e. we can refer the number $1-a$ as a spectral parameter.

Now we define the generalized solution in the space $W_\alpha^2(b)$.

Definition 3. The function $u \in W_\alpha^2(b)$ is called the generalized solution of the mixed problem for equation (1), if for every $v \in W_\alpha^2(b)$ the equality (8) holds.

Note that, if the generalized solution $u \in W_\alpha^2(b)$ is classical, then for $\alpha = 0$ we get the following conditions (see [5]) $u''(0) = u'''(0) = u(b) = u'(b) = 0$.

Consider the particular case of equation (1) for $a = 0$

$$Su \equiv \left(t^\alpha u''\right)'' = f. \tag{10}$$

Proposition 4. For every $f \in L_2(0,b)$ the generalized solution of the mixed problem for equation (10) exists and is unique.

The proof is similar to the proof of the Proposition 3.

Define the operator $S : L_2(0,b) \to L_2(0,b)$ corresponding to Definition 3.

Definition 4. We say that the function $u \in W_\alpha^2(b)$ belongs to the domain $D(S)$ of operator S, if there exists $f \in L_2(0,b)$, such that the equality (8) is valid. In this case we write $Su = f$.

Theorem 2. Operator $S : L_2(0,b) \to L_2(0,b)$ is positive and self-adjoint. The bounded operator $S^{-1} : L_2(0,b) \to L_2(0,b)$ for $0 \le \alpha < 4$ is compact.

The proof is similar to the proof of the Theorem 1.

Corollary 2. For $0 \le \alpha < 4$ the operator S has a discrete spectrum, and its eigenfunction system is complete in $L_2(0,b)$ (see [6]).

Note that now we can rewrite equation (1) in the form $Su = -au + f$, i.e. we can refer the number $-a$ as a spectral parameter.

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Էսմայիլ Յուսեֆի

Խառը խնդիր չորրորդ կարգի սովորական վերասերվող դիֆերենցիալ հավասարման համար

Աշխատանքում դիտարկվում է խառը խնդիր հետևյալ հավասարման համար.

$$Lu \equiv (t^{\alpha}u + '')''au = f, \quad (1)$$

որտեղ $0 \le \alpha \le 4$, $t \in [0,b], f \in L_2(0,b)$: Նախ սահմանվում են Սոբոլևի կշռային W_α^2, $W_\alpha^2(0)$, $W_\alpha^2(b)$ տարածությունները և (1) հավասարման ընդհանրացված լուծումը: Այնուհետև դիտարկվում են ընդհանրացված լուծման գոյության և միակության հարցերը, ինչպես նաև տրվում է համապատասխան օպերատորի սպեկտրի նկարագիրը:

Есмаил Юсефи

Смешанная задача для обыкновенного вырождающегося дифференциального уравнения четвертого порядка

В работе рассматривается смешанная задача для уравнения

$$Lu \equiv (t^{\alpha}u'')'' + au = f, \quad (1)$$

где $0 \le \alpha \le 4$, $t \in [0,b]$, $f \in L_2(0,b)$. Сперва определяются весовые пространства Соболева W_α^2, $W_\alpha^2(0)$, $W_\alpha^2(b)$ и обобщенное решение для уравнения (1). Затем изучается вопрос существования и единственности обобщенного решения, а также дается описание спектра для соответствующего оператора.