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Mathematics

INDEPENDENT PAIRS IN FREE BURNSIDE GROUPS

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In this work we prove that for an arbitrary odd $n \geq 1003$ there exist two words $u(x,y), v(x,y)$, almost every images of which in free Burnside group $B(m,n)$ are independent.

***Keywords*:** free Burnside group, independent element, non-amenable group, monomorphism.

1. Introduction. A free Burnside group $B(m,n)$ is defined as relatively free m-generated group of all groups variety that satisfy the identity $X^n = 1$. It has the following presentation:

$$B(m,n) = < a_1, \ldots, a_m \mid A^n = 1 \text{ for all words } A = A(a_1, \ldots, a_m) > .$$

Subgroups of groups $B(m,n)$ for the case of all odd $n \geq 665$ are studied in [1–6] and for odd $n > 10^{80}$ in [7–9].

Definition. Elements u and v of the group $B(m,n)$ are called independent, if they generate a subgroup isomorphic to $B(2,n)$.

In [10] it is proved that for an arbitrary sufficiently large odd number n ($n \geq 1039$) there exist two words $u(x,y), v(x,y)$, such that for some k the elements $u(a^k,b), v(a^k,b)$ are independent, where a,b are any two noncommuting elements of the free Burnside group $B(m,n)$. Such a hypothesis was formulated in 1989 in survey [11]. In this work we lower the bound for those n, for which this hypothesis is true.

2. *Theorem*. For arbitrary odd n, where $n \geq 1003$, there exist words $u(x,y), v(x,y)$ such that, if a,b are any two elements generating noncyclic subgroup of group $B(m,n)$, then for some p elements $u(a^p,b)$ and $v(a^p,b)$ are independent.

In the proof of Theorem the work [12] is used, where the inequality $n \geq 1003$ first meets in study of periodic groups.

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Words $u(x,y)$ and $v(x,y)$ whose existence is stated in the Theorem are defined in the following way: let $w(x,y) \rightleftharpoons [x, yxy^{-1}]$ and $W(x,y) \rightleftharpoons$ $\rightleftharpoons [w(x,y)^d, xw(x,y)^d x^{-1}]$, where $d = 191$. As words $u(x,y)$ and $v(x,y)$ choose words

$$u(x,y) \rightleftharpoons W(x,y)^{200} w(x,y) W(x,y)^{200} w(x,y)^2 ... W(x,y)^{200} w(x,y)^{n-1} W(x,y)^{200}, \quad (1)$$

$$v(x,y) \rightleftharpoons W(x,y)^{300} w(x,y) W(x,y)^{300} w(x,y)^2 ... W(x,y)^{300} w(x,y)^{n-1} W(x,y)^{300}. \quad (2)$$

Denoting $u_k(x,y) = u(x^{2^k}, y)$, $v_k(x,y) = v(x^{2^k}, y)$ and comparing Theorem with the result of paper [8], we obtain

Corollary 1. For arbitrary odd $n \geq 1003$, if a and b are any two noncommuting elements of group $B(m,n)$, then one of the pairs of words $\{u_0(a,b), v_0(a,b)\}, ..., \{u_9(a,b), v_9(a,b)\}$ is independent.

Let $\phi : B(2,n) \to B(2,n)$ be a homomorphism of group $B(m,n)$, defined on free generating elements x and y by formulas $\phi(x) = u(x,y)$, $\phi(y) = v(x,y)$. It is clear that homomorphism $\tau : B(2,n) \to B(2,n)$, $\tau(x) = x^{2^k}$, $\tau(y) = y$ is an automorphism, since it obviously has an inverse. From Theorem immediately follows, that for some k the composition $\tau \circ \phi$ is a monomorphism. Therefore, ϕ is a monomorphism as well. Using these monomorphisms we get, that if $\varphi(x) = a, \varphi(y) = b$, then

$$\varphi \circ \tau \circ \phi(x) = \varphi(\tau(\phi(x))) = u(a^{2^k}, b), \varphi \circ \tau \circ \phi(y) = \varphi(\tau(\phi(y))) = v(a^{2^k}, b).$$

Thus, the following corollary holds.

Corollary 2. There exists a monomorphism $\phi : B(2,n) \to B(2,n)$, such that for any endomorphism $\varphi : B(2,n) \to B(2,n)$ with noncyclic image there exists an automorphism $\tau : B(2,n) \to B(2,n)$, such that $\varphi \circ \tau \circ \phi$ is a monomorphism.

One of classical results of S.I.Adian states, that for any odd $n \geq 665$ and $m > 1$ free Burnside groups $B(m,n)$ are non-amenable (see [13]).

From Theorem and Corollary 2 of paper [10] follows

Corollary 3. For arbitrary odd $n \geq 1003$ the group $B(m,n)$ is uniformly non-amenable.

Uniform non-amenable groups $B(m,n)$ and their subgroups are studied in [14–16].

3. Proof of Theorem.

Lemma 1. For arbitrary odd $n \geq 1003$ and any r, $1 \leq r \leq \frac{(n-1)}{2}$, there exist integers s and k, $186 \leq s \leq \frac{(n+1)}{2} - 148$, $0 \leq k \leq 9$, such that one of the following congruences $r \cdot 2^k \equiv s (\mathrm{mod}\, n)$ and $(-r) 2^k \equiv s (\mathrm{mod}\, n)$ holds.

Let n be $n \geq 1039$. For $186 \leq r \leq \frac{n+1}{2} - 148$ one can choose $k = 0$, and if $\frac{186}{2^k} \leq r \leq \frac{372}{2^k}$, where $k = 1,...,8$, then $186 \leq r \cdot 2^k \leq 372 \leq \frac{n+1}{2} - 148$ (since

$n \geq 1039$). But if $\frac{n+1}{2} - 148 \leq r \leq \frac{n-1}{2}$, then $1 \leq n - 2r \leq 295 \leq \frac{n+1}{2} - 148$, and one can use the above-mentioned reasoning. Thus, for some $p = 2^k$, where $0 \leq k \leq 9$, holds $r \cdot 2^k \equiv s(\mathrm{mod}\, n)$ or $(-r)2^k \equiv s(\mathrm{mod}\, n)$, where $186 \leq s \leq \frac{n+1}{2} - 148$.

Now let n and s be such that $1003 \leq n \leq 1039$, $186 \leq s \leq \frac{(n+1)}{2} - 148$.

If $186 \leq r \leq 354$, then in order to prove Lemma 1 it is enough to take $k = 0$ and $s = r$.

Now let $1 \leq r \leq 185$. Then we can:

1) for $178 \leq r \leq 185$ take $k = 2$ and $s = n - 2^k \cdot r$;
2) for $93 \leq r \leq 177$ take $k = 1$ and $s = 2^k \cdot r$;
3) for $89 \leq r \leq 92$ take $k = 3$ and $s = n - 2^k \cdot r$;
4) for $47 \leq r \leq 88$ take $k = 2$ and $s = 2^k \cdot r$;
5) for $45 \leq r \leq 46$ take $k = 4$ and $s = n - 2^k \cdot r$;
6) for $24 \leq r \leq 44$ take $k = 3$ and $s = 2^k \cdot r$;
7) for $r = 23$ take $k = 5$ and $s = n - 2^k \cdot r$;
8) for $12 \leq r \leq 22$ take $k = 4$ and $s = 2^k \cdot r$;
9) for $6 \leq r \leq 11$ take $k = 5$ and $s = 2^k \cdot r$;
10) for $3 \leq r \leq 5$ take $k = 6$ and $s = 2^k \cdot r$;
11) for $r = 2$ take $k = 7$ and $s = 2^k \cdot r$;
12) for $r = 1$ take $k = 8$ and $s = 2^k$.

Thus, for any r, $1 \leq r \leq 354$, there exist k and s, $0 \leq k \leq 8$, $186 \leq s \leq \frac{(n+1)}{2} - 148$, such that either $s \equiv r \cdot 2^k (\mathrm{mod}\, n)$ or $s \equiv (-r)2^k (\mathrm{mod}\, n)$ holds. It remains to consider the case $354 \leq r \leq \frac{n-1}{2}$. By denoting $r_1 \rightleftharpoons n - 2r$ and putting $k = 1$ we get $1 \leq r_1 \leq 354$, whereas $-2r \equiv r_1 (\mathrm{mod}\, n)$. Due to the case considered above, for r_1 there exist k_1 and s_1, such that either $s_1 \equiv r_1 \cdot 2^{k_1} (\mathrm{mod}\, n)$ or $s_1 \equiv (-r_1)2^{k_1} (\mathrm{mod}\, n)$ holds, where $0 \leq k_1 \leq 8$.

Comparing all the cases, we finally conclude that $0 \leq k \leq 9$. Lemma 1 is proved. The following Lemma is proved in [10].

Lemma 2. (see Lemma 2 [10]). Suppose n is an arbitrary odd number $n \geq 665$. If a and b do not commute in $B(m,n)$ and $a^p \neq 1$, then $w(a^p, b) \neq 1$.

Lemma 3. (compare with Lemma 3 [10]). Suppose n is an arbitrary odd number $n \geq 1003$. If a and b do not commute in $B(m,n)$ and a is a conjugate element to power of some elementary period E of rank γ, then for some $p = 2^k$, $0 \leq k \leq 9$, the element $w(a^p, b)$ is a conjugate element to some elementary period of rank $\beta \geq \gamma + 1$.

Proof. Suppose that for some word T we have $a = TE^rT^{-1}$ in $B(m,n)$. Replacing, if necessary, E with E^{-1} we can assume that $1 \le r \le \frac{n-1}{2}$. According to Lemma 1, for some $p = 2^k$, where $0 \le k \le 9$, we have $a^p = TE^{rp}T^{-1} = TE^sT^{-1}$ and the inequality $186 \le s \le \frac{n+1}{2} - 148$ holds. Due to Lemma 2.8 [12], we may choose the period E minimized, and due to VI.2.4 and IV.3.12 [1] one can assume that $T^{-1}bT \in \mathcal{M}_\gamma \cap \mathcal{A}_{\gamma+1}$. According to Lemma 2, we have $T^{-1}w(a^p,b)T \ne 1$ in the group $B(m,n)$, therefore, $[E^s, T^{-1}bTE^sT^{-1}b^{-1}T] \ne 1$, and due to Lemma 3.2 [12], one can indicate the reduced form A of commutator $[E^s, T^{-1}bTE^sT^{-1}b^{-1}T]$ which, according to Lemma 7.2 [12], is an elementary period of some rank $\beta \ge \gamma + 1$.

The following two Lemmas are proved in [10].

Lemma 4. (see Lemma 4[10]). Suppose n is arbitrary odd number $n \ge 1003$. Assume that a and b do not commute in $B(m,n)$, element a is a conjugate element to power of some elementary period E of rank γ, and for some p the element $w(a^p,b)$ is a conjugate element to some elementary period of rank $\beta \ge \gamma + 1$. Then $W(a^p,b) \ne 1$ in $B(m,n)$.

Lemma 5. (see Lemma 5[10]). Suppose n is arbitrary odd number $n \ge 1003$ and a and b are two noncommuting elements of $B(2,n)$. Then for some $p = 2^k, 0 \le k \le 9$, words $u(a^p,b), v(a^p,b)$ freely generate a free Burnside subgroup of group $B(2,n)$, where words $u(x,y)$ and $v(x,y)$ are defined by equalities (1) and (2).

Lemma 6. Suppose n is arbitrary odd number $n \ge 1003$, a and b are two noncommuting elements of group $B(2,n)$. Then for some $p = 2^k$ with $0 \le k \le 9$, words $u(a^p,b)$ and $v(a^p,b)$ are independent, where words $u(x,y)$ and $v(x,y)$ are defined by relations (1) and (2).

Proof. It is necessary to repeat the proof of Lemma 5 [10], changing the reference to Lemma 3 of [10] by reference to Lemma 3 of the current work.

Proof of Theorem. From Theorem VI.3.7 [1] by S.I.Adian immediately follows, that for arbitrary odd n>665 and finite m the group $B(m,n)$ can be isomorphically embedded into group $B(2,n)$. Therefore, it is enough to prove the Theorem for the case $m = 2$. But in this case the validity of Theorem follows from Lemma 6.

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REFERENCES

1. **Adian S.I.** The Burnside Problem and Identities in Groups. Ergeb. Math. Grenzgeb., 95. Berlin–Heidelberg–New York: Springer–Verlag, 1979.
2. **Adian S.I.** Trudy Mat. Inst. Steklova, 1971, v. 112, p. 61–69 (in Russian).

3. **Shirvanian V.L.** Izvestia AN SSSR. Matematica, 1976, v. 10, № 1, p. 181–199 (in Russian).
4. **Adian S.I.** Izvestia AN SSSR. Matematica, 1982, v. 19, № 2, p. 215–229 (in Russian).
5. **Atabekian V.S.** Izvestia AN SSSR. Matematica, 2009, v. 73, № 5, p. 861–892 (in Russian).
6. **Atabekian V.S.** Prikladnaya Matematica, 2009, v. 15, № 1, p. 3–21(in Russian).
7. **Olshanskii A.Yu.** Groups, Rings, Lie and Hopf Algebras. Kluwer AP, 2003, p. 179–187.
8. **Ivanov S.V.** Illinois J. Math., 2003, v. 47, № 1–2, p. 299–304.
9. **Sonkin D.** Comm. Algebra, 2003, v. 31, № 10, p. 4687–4695.
10. **Atabekian V.S.** Mat. Zametki, 2009, v. 86, № 4, p. 457–462 (in Russian).
11. **Ivanov S.V., Ol'shanskii A.Yu.** St. Andrews, 1989, v. 2. London Math. Soc. Lecture Note Ser., Cambridge: Cambridge Univ. Press, 1991, v. 160, p. 258–308.
12. **Adyan S.I., Lysenok I.G.** Izvestia AN SSSR. Matematica, 1992, v. 39, № 2, p. 905–957 (in Russian).
13. **Adian S.I.** Izvestia AN SSSR. Matematica, 1983, v. 21, № 3, p. 425–434 (in Russian).
14. **Arzhantseva G.N., Burillo J., Lustig M., Reeves L., Short H., Ventura E.** Adv. Math., 2005, v. 197, № 2, p. 499–522.
15. **Osin D.V.** Arch. Math. (Basel), 2008, v. 88, № 5, p. 403–412.
16. **Atabekian V.S.** Mat. Zametki, 2009, v. 85, № 4, p. 496–502 (in Russian).

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ԱՆԿԱԽ ԶՈՒՅԳԵՐ ԱԶԱՏ ԲԵՌՆՍԱՅԴՅԱՆ ԽՄԲԵՐՈՒՄ

Աշխատանքում ապացուցվում է, որ կամայական $n \geq 1003$ կենտ թվի համար գոյություն ունեն երկու բառեր՝ $u(x, y), v(x, y)$, այնպիսիք, որ ազատ բեռնսայդյան խմբում նրանց համարյա բոլոր պատկերներն անկախ են:

А. С. ПАЙЛЕВАНЯН

НЕЗАВИСИМЫЕ ПАРЫ В СВОБОДНЫХ БЕРНСАЙДОВЫХ ГРУППАХ

В работе доказывается, что для произвольного нечетного $n \geq 1003$ существуют два слова $u(x, y), v(x, y)$, почти все образы в свободной бернсайдовой группе $B(m,n)$ которых независимы.