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Mathematics

ON THE BOUNDEDNESS OF A CLASS OF THE FIRST ORDER LINEAR DIFFERENTIAL OPERATORS

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In the present article the first order linear differential operators with unbounded coefficients are investigated. The boundedness of the operators under consideration was proved.

***Keywords*:** bounded differential operator, first order differential operator.

Let $Q \subset R^n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial Q \in C^1$. Consider the first order differential expression

$$Tu \equiv (\bar{b}(x), \nabla u(x)) - div(\bar{c}(x)u(x)) + d(x)u(x), \quad u \in \overset{\circ}{W}{}_2^1(Q),$$

with coefficients $\bar{b}(x) = (b^{(1)}(x), \ldots, b^{(n)}(x))$, $\bar{c}(x) = (c^{(1)}(x), \ldots, c^{(n)}(x))$ and $d(x)$ that are measurable and bounded on each strong inner subdomain of the domain Q.

For an arbitrary $u, v \in \overset{\circ}{W}{}_2^1(Q)$ define

$$\langle Tu, v \rangle \equiv \int_Q ((\bar{b}(x), \nabla u(x))v(x) + (\bar{c}(x)u(x), \nabla v(x)) + d(x)u(x)v(x))dx.$$

The aim of this article is to obtain conditions to be imposed on the coefficients $\bar{b}(x), \bar{c}(x)$ and $d(x)$, for which T is a linear bounded operator acting from $\overset{\circ}{W}{}_2^1(Q)$ into $\overset{\circ}{W}{}_2^{-1}(Q)$. This property has important applications in studying the problems of mathematical physics (see, for example, [1, 2]).

The following theorem is proved.

Theorem. Let the following conditions hold

$$\left|\bar{b}(x)\right| = O\left(\frac{1}{r(x)}\right) \text{ as } r(x) \to 0, \tag{1}$$

where $r(x)$ is the distance of a point $x \in Q$ from the boundary ∂Q,

$$\int_0 tC^2(t)dt < \infty, \text{ where } C(t) = \sup_{r(x) \geq t} \left|\bar{c}(x)\right|, \tag{2}$$

$$\int_0 t^3 D^2(t)dt < \infty, \text{ where } D(t) = \sup_{r(x) \geq t} \left|d(x)\right|. \tag{3}$$

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Then the operator T is a bounded linear operator from $\overset{\circ}{W}{}_2^1(Q)$ into $\overset{\circ}{W}{}_2^{-1}(Q)$.

Proof of Theorem. Let $x^0 \in \partial Q$ be an arbitrary point of the boundary ∂Q of the domain Q, (x', x_n) be a local coordinate system with the origin x^0 and the x_n axis directed along the inner normal $\nu(x^0)$ to ∂Q at the point x^0. Since $\partial Q \in C^1$, there exists a positive number $r_{x^0} > 0$ and a function $\varphi_{x^0} \in C^1(R^{n-1})$ with properties

$$\varphi_{x^0}(0) = 0,\ \nabla\varphi_{x^0}(0) = 0 \text{ and } \left|\nabla\varphi_{x^0}(x')\right| \le \frac{1}{2} \text{ for all } x' \in R^{n-1},$$

such that the intersection of the domain Q with the ball $U_{x^0}^{(r_{x^0})} = \{x : \left|x - x^0\right| < r_{x^0}\}$ of radius r_{x^0} and the centre x^0 has the form $Q \cap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \cap \{(x', x_n) : x_n > \varphi_{x^0}(x')\}$. Then $\partial Q \cap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \cap \{(x', x_n) : x_n = \varphi_{x^0}(x')\}$. Let $l_{x^0} = \frac{r_{x^0}}{\sqrt{2}}$. From the covering $\{U_{x^0}^{(l_{x^0})}, x^0 \in \partial Q\}$ of the boundary ∂Q select a finite subcovering $U_{x^m}^{(l_{x^m})}$, $m = 1,\dots,p$. Denote for simplicity $U_{x^m}^{(l_{x^m})}$ by U_m, r_{x^m} by r_m, l_{x^m} by l_m, φ_{x^m} by φ_m, $m = 1,\dots,p$. Now set $h = \frac{1}{3}\left(\frac{2}{\sqrt{5}} - \frac{\sqrt{2}}{2}\right)\min(r_1,\dots,r_p)$. Then each of the curvilinear “cylinders “

$$\Pi_m^{l_m,h} = \{(x', x_n) : \left|x'\right| < l_m, \varphi_m(x') < x_n < \varphi_m(x') + h\},\ m = 1,\dots,p,$$

is contained in the corresponding ball U_m, as well as in $U_m \cap Q$ (recall that (x', x_n) are the coordinates of a point in a local system of coordinates with origin at x^m). Let $l_0 < h$ be a positive number such that the complement of the domain $Q_{l_0} = \{x \in Q : r(x) = \operatorname{dist}(x, \partial Q) > l_0\}$ in Q is contained in the union of the ”cylinders” $\Pi_m^{l_m,h}$, $m = 1,\dots,p$, i.e. $Q^{l_0} = \{x \in Q : r(x) = \operatorname{dist}(x, \partial Q) \le l_0\} \subset \bigcup_{m=1}^{p} \Pi_m^{l_m,h}$.

It is easily verified that for all $x = (x', x_n) \in \Pi_m^{l_m,h}$, $m = 1,\dots,p$,

$$r(x) \le x_n - \varphi_m(x') \le \frac{\sqrt{5}}{2} r(x).$$

Now fix some number m, $1 \le m \le p$, and take a local coordinate system with origin at x^m.

We define mappings L and L_{-1} of the space R^n onto itself using relations $L(x) = (x', x_n - \varphi_m(x'))$, where $x = (x', x_n)$ and $L_{-1}(y) = (y', y_n + \varphi_m(y'))$, $y = (y', y_n)$. The image of $\Pi_m^{l_m,h}$ under the mapping L will be denoted by $\tilde{\Pi}_m^{l_m,h}$:

$$L\left(\Pi_m^{l_m,h}\right) = \tilde{\Pi}_m^{l_m,h}.$$

Now take arbitrary functions $u \in \overset{\circ}{W}{}_2^1(Q)$ and $\eta \in C_0^\infty(Q)$ and make the notations $u(y', y_n + \varphi(y')) = \tilde{u}(y)$, $\eta(y', y_n + \varphi(y')) = \tilde{\eta}(y)$.

In view of (1), (2) and (3) we have

$$\left|\langle Tu,\eta\rangle\right| \le K\int_Q \frac{|\nabla u(x)||\eta(x)|}{r(x)}dx + \int_Q C(r(x))|u(x)||\nabla\eta(x)|\,dx + \int_Q D(r(x))|u(x)||\eta(x)|\,dx\,,$$

where K is a constant.

Let us estimate

$$I_1 = \int_Q \frac{|\nabla u(x)||\eta(x)|}{r(x)}dx \le \int_{Q^{l_0}} \frac{|\nabla u(x)||\eta(x)|}{r(x)}dx + \frac{1}{l_0}\int_{Q_{l_0}} |\nabla u(x)||\eta(x)|\,dx \le$$

$$\le \sum_{m=1}^{p}\int_{\Pi_m^{l_m,h}} \frac{|\nabla u(x)||\eta(x)|}{r(x)}dx + \frac{1}{l_0}\|u\|_{\overset{\circ}{W}{}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\,.$$

For $m = 1,\dots,p$ the following estimate holds:

$$\int_{\Pi_m^{l_m,h}} \frac{|\nabla u(x)||\eta(x)|}{r(x)}dx \le \sqrt{\frac{5}{2}}\int_{\tilde\Pi_m^{l_m,h}} \frac{|\nabla\tilde u(y)||\tilde\eta(y)|}{y_n}dy \le \sqrt{\frac{5}{2}}\left(\int_{\tilde\Pi_m^{l_m,h}} |\nabla\tilde u(y)|^2 dy\right)^{1/2}\left(\int_{\tilde\Pi_m^{l_m,h}} \frac{\tilde\eta^2(y)}{y_n^2}dy\right)^{1/2} \le$$

$$\le \sqrt{5}\left(\int_{\Pi_m^{l_m,h}} |\nabla u(x)|^2 dx\right)^{1/2}\left(\int_{\tilde\Pi_m^{l_m,h}} \frac{\tilde\eta^2(y)}{y_n^2}dy\right)^{1/2} \le \mathrm{const}\,\|u\|_{\overset{\circ}{W}{}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\,.$$

We used here the Hardy inequality (see, for example, [3]), in virtue of which

$$\int_{\tilde\Pi_m^{l_m,h}} \frac{\tilde\eta^2(y)}{y_n^2}dy = \int_0^h dy_n \int_{|y'|<l_m} dy'\,\frac{\tilde\eta^2(y',y_n)}{y_n^2} \le \mathrm{const}\int_{\tilde\Pi_m^{l_m,h}} |\nabla\tilde\eta(y)|^2\,dy\,.$$

Thus,

$$I_1 \le \mathrm{const}\,\|u\|_{\overset{\circ}{W}{}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\,, \tag{4}$$

where the constant does not depend on u and η. Next,

$$I_2 = \int_Q C(r(x))|u(x)||\nabla\eta(x)|\,dx \le \int_{Q^{l_0}} C(r(x))|u(x)||\nabla\eta(x)|\,dx + C(l_0)\int_{Q_{l_0}} |u(x)||\nabla\eta(x)|\,dx \le$$

$$\le \sum_{m=1}^{p}\int_{\Pi_m^{l_m,h}} C(r(x))|u(x)||\nabla\eta(x)|\,dx + C(l_0)\|u\|_{\overset{\circ}{W}{}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\,.$$

For $m = 1,\dots,p$ we have

$$\int_{\Pi_m^{l_m,h}} C(r(x))|u(x)||\nabla\eta(x)|\,dx \le \left(\int_{\Pi_m^{l_m,h}} C^2(r(x))u^2(x)dx\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} \le$$

$$\le \left(\int_{\tilde\Pi_m^{l_m,h}} C^2\left(\frac{2}{\sqrt5}y_n\right)\tilde u^2(y)dy\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} \le \left(\int_{\tilde\Pi_m^{l_m,h}} C^2\left(\frac{2}{\sqrt5}y_n\right)y_n\int_0^{y_n}|\nabla\tilde u(y',\tau)|^2 d\tau dy\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} \le$$

$$\le \left(\int_0^h dy_n C^2\left(\frac{2}{\sqrt5}y_n\right)y_n\int_{|y'|<l_m} dy'\int_0^h d\tau\,|\nabla\tilde u(y',\tau)|^2\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} \le$$

$$\le \sqrt2\left(\int_0^h C^2\left(\frac{2}{\sqrt5}y_n\right)y_n dy_n\right)^{1/2}\|u\|_{\overset{\circ}{W}{}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\,.$$

Thus, we obtain

$$I_2 \le \text{const}\|u\|_{\overset{\circ}{W}{}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}, \tag{5}$$

where the constant does not depend on u and η.

Similarly we obtain

$$I_3 = \int_Q D(r(x))|u(x)||\eta(x)|dx \le \int_{Q^{l_0}} D(r(x))|u(x)||\eta(x)|dx + D(l_0)\int_{Q_{l_0}} |u(x)||\eta(x)|dx \le$$

$$\le \sum_{m=1}^{p} \int_{\Pi_m^{l_m,h}} D(r(x))|u(x)||\eta(x)|dx + D(l_0)\|u\|_{\overset{\circ}{W}{}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}.$$

Finally, for $m = 1,\dots,p$ we have

$$\int_{\Pi_m^{l_m,h}} D(r(x))|u(x)||\eta(x)|dx \le \int_{\tilde{\Pi}_m^{l_m,h}} D\left(\frac{2}{\sqrt{5}}y_n\right)|\tilde{u}(y)||\tilde{\eta}(y)|dy \le$$

$$\le \left(\int_{\tilde{\Pi}_m^{l_m,h}} D^2\left(\frac{2}{\sqrt{5}}y_n\right)y_n^2\tilde{u}^2(y)dy\right)^{1/2}\left(\int_{\tilde{\Pi}_m^{l_m,h}} \frac{\tilde{\eta}^2(y)}{y_n^2}dy\right)^{1/2} \le$$

$$\le \text{const}\left(\int_{\tilde{\Pi}_m^{l_m,h}} D^2\left(\frac{2}{\sqrt{5}}y_n\right)y_n^3\int_0^{y_n}|\nabla\tilde{u}(y',\tau)|^2 d\tau dy\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} \le$$

$$\le \text{const}\left(\int_0^h D^2\left(\frac{2}{\sqrt{5}}y_n\right)y_n^3 dy_n\right)^{1/2}\|u\|_{\overset{\circ}{W}{}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}.$$

Thus,

$$I_3 \le \text{const}\|u\|_{\overset{\circ}{W}{}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}, \tag{6}$$

where the constant is independent of u and η.

Therefore, in view of (4)–(6) the following estimate holds $|\langle Tu,\eta\rangle| \le \text{const}\|u\|_{\overset{\circ}{W}{}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}$, where the constant is independent of u and η.

Since the functions $\eta(x)$ from $C_0^\infty(Q)$ are dense everywhere in $\overset{\circ}{W}{}_2^1(Q)$, the proof of the Theorem immediately follows from the established estimate. The Theorem is proved.

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