PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2011, № 1, p. 7–11

Mathematics

ON THE ANISOTROPIC BOUNDARY VALUE PROBLEM, CONNECTED WITH HELMHOLTZE-SCHRÖDINGER EQUATION, UNDER THE BOUNDARY CONDITIONS OF THE FIRST AND SECOND TYPE

S. A. HOSSEINY MATIKOLAI *

Chair of Differential Equations and Functional Analysis, YSU

In this paper we consider the solvability of the boundary value problem, connected with the anisotropic Helmholtz-Schrödinger equation, under the boundary conditions of the first and second type on the line $y = 0$.

***Keywords*:** Helmholtz-Shrödinger equation, factorization of matrix-function.

Introduction. The issue of solvability was considered in [1] for a class of boundary value problems, coordinated with the anisotropic Helmholtz-Shrödinger equation in the Sobolev space on the upper and lower half-planes, where the boundary conditions of the first and second type were fulfilled on $y = 0$ line. In [1] it was shown that the solvability of this problem is equivalent to that of some Riemann-Hilbert problem.

Let $\Omega^{\pm} = \left\{(x, y) \in R^2 : y > 0\,(y < 0)\right\}$ and $H^{1/2}\left(\Omega^{\pm}\right)$, $H^{-1/2}\left(\Omega^{\pm}\right)$ are the corresponding Sobolev spaces (see [2]). Now consider the following anisotropic Helmholtz-Schrödinger equation

$$\begin{cases} \Delta u + (k^2 + 2\beta_+^2 \sec h^2(\beta_+ y))u = 0 & \text{in } \Omega^+, \\ \Delta u + (k^2 + 2\beta_-^2 \sec h^2(\beta_- y))u = 0 & \text{in } \Omega^-. \end{cases} \tag{1}$$

This equation a particular case of equation (1) from [1], where $\operatorname{Re} k > 0, \operatorname{Im} k > 0$. Note that similar problems in the isotropic case were investigated in [3–6]. We suppose that the following boundary conditions are fulfilled:

$$\begin{cases} \begin{cases} a_0 u(x,+0) + b_0 u(x,-0) = h_0(x), \\ a_1 \dfrac{\partial u(x,+0)}{\partial y} + b_1 \dfrac{\partial u(x,-0)}{\partial y} = h_1(x) \end{cases} & \text{in } R^+, \\ \begin{cases} c_0 u(x,+0) + d_0 u(x,-0) = p_0(x), \\ c_1 \dfrac{\partial u(x,+0)}{\partial y} + d_1 \dfrac{\partial u(x,-0)}{\partial y} = p_1(x) \end{cases} & \text{in } R^-, \end{cases} \tag{2}$$

* E-mail: seyedalim895@yahoo.com

where $h_0 \in H^{1/2}(R^+)$, $h_1 \in H^{-1/2}(R^+)$, $p_0 \in H^{1/2}(R^-)$, $p_1 \in H^{-1/2}(R^-)$, the coefficients a_0, a_1, b_0, b_1, c_0, c_1, d_o, d_1 are the complex constants and $a_1 d_1 = b_1 c_1$.

Introduce the functions

$$\begin{cases} u_-(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{0} \left[a_0 u(x;+0) + b_0 u(x;-0) - h_0(x) \right] e^{i\lambda x} dx, \\ w_-(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{0} \left[a_1 \dfrac{\partial u(x;+0)}{\partial y} + b_1 \dfrac{\partial u(x;-0)}{\partial y} - h_1(x) \right] e^{i\lambda x} dx, \\ u_+(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{0}^{\infty} \left[c_0 u(x;+0) + d_0 u(x;-0) - p_0(x) \right] e^{i\lambda x} dx, \\ w_+(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{0}^{\infty} \left[c_1 \dfrac{\partial u(x;+0)}{\partial y} + d_1 \dfrac{\partial u(x;-0)}{\partial y} - p_1(x) \right] e^{i\lambda x} dx, \end{cases} \tag{3}$$

$$\begin{cases} m_1(\lambda) = \dfrac{\hat{h}_0(\lambda)}{\Delta(\lambda)} \left\{ a_1 d_0 \dfrac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)} + b_1 c_0 \dfrac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)} \right\} + \dfrac{a_0 d_0 - b_0 c_0}{\Delta(\lambda)} \hat{h}_1(\lambda) - \hat{p}_0(\lambda), \\ m_2(\lambda) = \dfrac{\hat{h}_1(\lambda)}{\Delta(\lambda)} \left\{ b_0 c_1 \dfrac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)} + a_0 d_1 \dfrac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)} \right\} - \hat{p}_1(\lambda) \end{cases} \tag{4}$$

and

$$\begin{cases} a(\lambda) = \dfrac{1}{\Delta(\lambda)} \left\{ b_1 \dfrac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)} \left(u_-(\lambda) + \hat{h}_0(\lambda) \right) - b_0 \left(w_-(\lambda) + \hat{h}_1(\lambda) \right) \right\}, \\ b(\lambda) = \dfrac{1}{\Delta(\lambda)} \left\{ a_1 \dfrac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)} \left(u_-(\lambda) + \hat{h}_0(\lambda) \right) + a_0 \left(w_-(\lambda) + \hat{h}_1(\lambda) \right) \right\}, \end{cases} \tag{5}$$

where

$$\hat{h}_0(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} h_0(x) e^{i\lambda x} dx, \quad \hat{h}_1(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} h_1(x) e^{i\lambda x} dx,$$

$$\hat{p}_0(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} p_0(x) e^{i\lambda x} dx, \quad \hat{p}_1(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} p_1(x) e^{i\lambda x} dx$$

and

$$\Delta(\lambda) = a_0 b_1 \frac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)} + a_1 b_0 \frac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)}.$$

As was noted above, the solvability of the boundary value problem (1)–(2) is equivalent to the solvability of the Riemann-Hilbert problem (see [1])

$$\vec{u}_+(\lambda) = L(\lambda) \vec{u}_-(\lambda) + \vec{m}(\lambda), \tag{6}$$

where the vector-functions

$$\vec{u}_+(\lambda) = \begin{pmatrix} u_+(\lambda) \\ w_+(\lambda) \end{pmatrix}, \quad \vec{u}_-(\lambda) = \begin{pmatrix} u_-(\lambda) \\ w_-(\lambda) \end{pmatrix}, \quad \vec{m}(\lambda) = \begin{pmatrix} m_1(\lambda) \\ m_2(\lambda) \end{pmatrix} \tag{7}$$

and the matrix-function

$$L(\lambda)=\begin{pmatrix} \dfrac{a_1d_0\left(\gamma^2(\lambda)-\beta_+^2\right)+b_1c_0\left(\gamma^2(\lambda)-\beta_-^2\right)}{a_1b_0\left(\gamma^2(\lambda)-\beta_+^2\right)+a_0b_1\left(\gamma^2(\lambda)-\beta_-^2\right)} & \dfrac{(a_0d_0-b_0c_0)\gamma(\lambda)}{a_1b_0\left(\gamma^2(\lambda)-\beta_+^2\right)+a_0b_1\left(\gamma^2(\lambda)-\beta_-^2\right)} \\ 0 & \dfrac{b_0c_1\left(\gamma^2(\lambda)-\beta_+^2\right)+a_0d_1\left(\gamma^2(\lambda)-\beta_-^2\right)}{a_1b_0\left(\gamma^2(\lambda)-\beta_+^2\right)+a_0b_1\left(\gamma^2(\lambda)-\beta_-^2\right)} \end{pmatrix}. \quad (8)$$

Consider the matrices $J_i=\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $i=0,1$, generated from the coefficients of the boundary conditions (2). It is easy to verify that without the loss of the generality the non-degenerate cases (i.e. $\Delta(\lambda)\neq 0, \det\left(L(\lambda)\right)\neq 0$) are possible only in the following cases:

1) $J_0=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $J_1=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, 2) $J_0=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $J_1=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,

3) $J_0=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $J_1=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, 4) $J_0=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $J_1=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$,

5) $J_0=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J_1=\begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}$, 6) $J_0=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $J_1=\begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}$,

7) $J_0=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $J_1=\begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}$, 8) $J_0=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $J_1=\begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}$,

9) $J_0=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $J_1=\begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}$, 10) $J_0=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $J_1=\begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}$,

where $\nu\neq 0,1$ is a constant.

In all cases 1)–10) the matrix-function $L(\lambda)$ can be presented in the form

$$L(\lambda)=\begin{pmatrix} \dfrac{\alpha\gamma^2(\lambda)-\chi}{\delta\gamma^2(\lambda)-\rho} & \dfrac{\varepsilon\gamma(\lambda)}{\delta\gamma^2(\lambda)-\rho} \\ 0 & 1 \end{pmatrix}, \quad (9)$$

where in the case

1) $\alpha=\delta=1,\ \chi=\rho=\beta_+^2,\ \varepsilon=1,$ 6) $\alpha=\nu,\ \delta=\nu+1,\ \chi=\beta_+^2,\ \rho=\beta_-^2+\nu\beta_+^2,\ \varepsilon=1,$

2) $\alpha=\delta=1,\ \chi=\rho=\beta_+^2,\ \varepsilon=-1,$ 7) $\alpha=1,\ \delta=\nu+1,\ \chi=\beta_-^2,\ \rho=\beta_-^2+\nu\beta_+^2,\ \varepsilon=-1,$

3) $\alpha=\delta=1,\ \chi=\rho=\beta_-^2,\ \varepsilon=-1,$ 8) $\alpha=1,\ \delta=\nu,\ \chi=\beta_+^2,\ \rho=\nu\beta_+^2,\ \varepsilon=-1,$

4) $\alpha=\delta=1,\ \chi=\rho=\beta_-^2,\ \varepsilon=1,$ 9) $\alpha=\nu+1,\ \delta=\nu,\ \chi=\beta_-^2+\nu\beta_+^2,\ \rho=\nu\beta_+^2,\ \varepsilon=-1,$

5) $\alpha=\nu,\ \delta=1,\ \chi=\beta_+^2,\ \rho=\beta_-^2,\ \varepsilon=1,$ 10) $\alpha=\nu+1,\ \delta=\nu,\ \chi=\beta_-^2+\nu\beta_+^2,\ \rho=\beta_-^2,\ \varepsilon=-1.$

Recall that the generalized factorization of matrix $L(\lambda)$ in space $L^2(R)$ is the following representation:

$$L(\lambda)=L_+(\lambda)\begin{pmatrix} \left(\dfrac{\lambda-i}{\lambda+i}\right)^{\chi_1} & 0 \\ 0 & \left(\dfrac{\lambda-i}{\lambda+i}\right)^{\chi_2} \end{pmatrix}L_-(\lambda), \quad (10)$$

where

a) $\chi_1, \chi_2 \in Z$, $L_{\pm}^{\pm 1} \in L^2(R,\rho)$ (i.e. each component of the matrix belongs to $L^2(R,\rho)$), where $\rho(\lambda) = \dfrac{1}{\sqrt{\lambda^2+1}}$. The matrix-functions $L_{+}^{\pm 1}(\lambda)$ are analytically continued in the upper half-plane $\operatorname{Im}\lambda > 0$ and $L_{-}^{\pm 1}(\lambda)$ are analytically continued in the lower half-plane $\operatorname{Im}\lambda < 0$;

b) the components of the matrix-function $\rho(\lambda)L_{+}^{\pm 1}(\lambda)$ ($\rho(\lambda)L_{-}^{\pm 1}(\lambda)$) belong to the $L^2(R)$ and have analytic continuations in the upper half-plane $\operatorname{Im}\lambda > 0$ (respectively the lower half-plane $\operatorname{Im}\lambda < 0$).

The factorization is called canonical when $\chi_1 = \chi_2 = 0$. Let

$$R_{+}(\lambda) = \frac{\alpha\left(\lambda + \sqrt{k^2 + \dfrac{\chi}{\alpha}}\right)}{\delta\left(\lambda + \sqrt{k^2 + \dfrac{\rho}{\delta}}\right)}, \qquad R_{-}(\lambda) = \frac{\lambda - \sqrt{k^2 + \dfrac{\chi}{\alpha}}}{\lambda - \sqrt{k^2 + \dfrac{\rho}{\delta}}}. \tag{11}$$

It is easy to see that the components $\rho(\lambda)L_{\pm}^{\pm 1}(\lambda)$ belong to $L^2(R)$ and $\rho(\lambda)L_{+}^{\pm 1}(\lambda)$ ($\rho(\lambda)L_{-}^{\pm 1}(\lambda)$) have the analytic continuations in the upper half-plane $\operatorname{Im}\lambda > 0$ (the lower half-plane $\operatorname{Im}\lambda < 0$).

Denote by

$$r_{+}(\lambda) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{r(\xi)d\xi}{\xi - \lambda}, \qquad r_{-}(\lambda) = \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{r(\xi)d\xi}{\xi - \lambda},$$

where $$r(\lambda) = \frac{\varepsilon\sqrt{\lambda^2 - k^2}}{\alpha\left(\lambda + \sqrt{k^2 + \dfrac{\chi}{\alpha}}\right)\left(\lambda - \sqrt{k^2 + \dfrac{\rho}{\delta}}\right)}, \quad k_{*} < c < d < k_{**}.$$

Here $k_{*} = \max\left\{-\operatorname{Im}\sqrt{k^2 + \dfrac{\chi}{\alpha}}; -\operatorname{Im}\sqrt{k^2 + \dfrac{\rho}{\delta}}\right\}$, $k_{**} = \min\left\{\operatorname{Im}\sqrt{k^2 + \dfrac{\chi}{\alpha}}; \operatorname{Im}\sqrt{k^2 + \dfrac{\rho}{\delta}}\right\}$.

It is evident that $r_{\pm}(\lambda)$ belong to the $L^2(R)$ and $r_{+}(\lambda)$ $(r_{-}(\lambda))$ have analytic continuation in the upper (lower) half-plane (see [7]). Since

$$R_{+}(\lambda)R_{-}(\lambda) = R(\lambda) = \frac{\alpha\gamma^2(\lambda) - \chi}{\delta\gamma^2(\lambda) - \rho}, \quad r_{+}(\lambda) + r_{-}(\lambda) = r(\lambda) \tag{12}$$

and

$$\begin{pmatrix} 1 & r_{\pm}(\lambda) \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -r_{\pm}(\lambda) \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} R_{\pm}(\lambda) & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{1}{R_{\pm}(\lambda)} & 0 \\ 0 & 1 \end{pmatrix}, \tag{13}$$

therefore, we have the equality

$$L(\lambda) = \begin{pmatrix} R_{+}(\lambda) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & r_{+}(\lambda) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & r_{-}(\lambda) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R_{-}(\lambda) & 0 \\ 0 & 1 \end{pmatrix}. \tag{14}$$

The representation (14) is the canonical factorization of the matrix-function $L(\lambda)$.

Using equalities (13) and (14) we can write the Riemann-Hilbert problem (6) in the form

$$\begin{pmatrix} \frac{1}{R_+(\lambda)} & -r_+(\lambda) \\ 0 & 1 \end{pmatrix} \vec{u}_+(\lambda) = \begin{pmatrix} R_-(\lambda) & r_-(\lambda) \\ 0 & 1 \end{pmatrix} \vec{u}_-(\lambda) + \begin{pmatrix} \frac{1}{R_+(\lambda)} & -r_+(\lambda) \\ 0 & 1 \end{pmatrix} \vec{m}(\lambda). \quad (15)$$

In an expanded form we get the following system:

$$\begin{cases} \frac{u_+(\lambda)}{R_+(\lambda)} - r_+(\lambda) w_+(\lambda) = R_-(\lambda) u_-(\lambda) + r_-(\lambda) w_-(\lambda) + \frac{m_1(\lambda)}{R_+(\lambda)} - r_+(\lambda) m_2(\lambda), \\ w_+(\lambda) = w_-(\lambda) + m_2(\lambda). \end{cases} \quad (16)$$

Hence we have

$$w_+(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} m_2(x) e^{i\lambda x} dx, \qquad w_-(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} m_2(x) e^{i\lambda x} dx,$$

$$u_+(\lambda) = \frac{r_+(\lambda) R_+(\lambda)}{\sqrt{2\pi}} \int_0^{\infty} m_2(x) e^{i\lambda x} dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} (m_1(x) + r_+(x) R_+(x) m_2(x)) e^{i\lambda x} dx,$$

$$u_-(\lambda) = \frac{r_-(\lambda)}{R_-(\lambda)\sqrt{2\pi}} \int_{-\infty}^{0} m_2(x) e^{i\lambda x} dx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \left(\frac{m_1(x)}{R_-(x)} + \frac{r_+(x) m_2(x)}{R_-(x)} \right) e^{i\lambda x} dx.$$

Using (5) and the following representation

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ a(\lambda) \frac{\gamma(\lambda) + \beta_+ \tanh(\beta_+ y)}{\gamma(\lambda)} e^{-\gamma(\lambda) y} \chi_+(y) + \right.$$
$$\left. + b(\lambda) \frac{\gamma(\lambda) - \beta_- \tanh(\beta_- y)}{\gamma(\lambda)} e^{\gamma(\lambda) y} \chi_-(y) \right\} e^{i\lambda x} d\lambda, \quad (17)$$

we obtain the solution of the boundary value problem (1)–(2).

Theorem. The boundary value problem (1)–(2) has the unique solution, which is given by the formula (17), where the functions $a(\lambda)$ and $b(\lambda)$ can be reconstructed by formula (5).

Received 21.07.2010

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