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# ON ONE NONLINEAR DIFFERENTIAL SEVERAL PERSON GAME IN CASE OF MANY AIM SETS

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The differential several person game in case of aim sets is considered, the dynamics of which is described by a nonlinear differential equation. A balance set of strategies is constructed by means of the method of extreme aiming at the corresponding set.

***Keywords*:** differential several person games, many aim sets, balance set of strategies.

**1.** A differential game is considered in the following statement. The dynamics of system is described by the differential equation

$$\dot{x} = f(t,x,u_1,...,u_k), \quad u_i \in P_i \subset R^{n_i}, \quad i = 1,2,...,k. \qquad (1.1)$$

Here $x \in R^n$ is a phase vector, $u_i$ is an operating influence of the $i$–th player, $P_i$ is a compact set in $R^{n_i}$ space, $f:[t_0,\infty)\times R^n \times R^{n_1} \times ... \times R^{n_k} \to R^n$ is the vector-function that is continuous on the set of all arguments at $t_0 \le t \le \theta$ ($t_0$ and $\theta$ are the given instants of time).

Let us introduce the following notations:

$$P = P_1 \times \ldots \times P_k, \; P^{(i)} = P_1 \times \ldots \times P_{i-1} \times P_{i+1} \times \ldots P_k,$$
$$u = (u_1,...,u_k), \; u^{(i)} = (u_1,...,u_{i-1},u_{i+1},...,u_k),$$
$$K = \{1,2,...,k\}, \; K(i) = \{1,...,i-1,i+1,...,k\}.$$

It is assumed that function $f(\cdot)$ satisfies the condition of infinite continuation of solutions, the condition of Lipschitz with respect to $x$ and there is a saddle point for the "small game" [1].

Let $\vartheta_j$ ($j = 1,\ldots,m$) be intermediate instants of time on $[t_0,\theta]$ interval such that $t_0 = \vartheta_0 < \vartheta_1 < \ldots < \vartheta_m = \theta$, and the compact sets $M_1^{(j)},...,M_k^{(j)}$ ($j = 1,...,m$) satisfy the following conditions: $M_i^{(j)} \cap G(\vartheta_j, t_0, x_0) \ne \varnothing$ ($i = 1,2,...,k$, $j = 1,...,m$).

Here the area of approachability of system (1.1) from the position $\{t_0, x_0\}$ at an instant $\vartheta_j$ is denoted as $G(\vartheta_j, t_0, x_0)$. The set $M_i^{(j)}$ is aim set for the $i$-th player at the instant $\vartheta_j$.

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Now consider the differential several person game for many aim sets, in which each player tries to reduce the total motion distance from the aim sets at the instants $\vartheta_j$, i.e. the aim of the $i$-th player is to approach the sets $M_i^{(j)}$ at instants $\vartheta_j$ ($j=1,\dots,m$).

In this work the questions of existence of balance set of strategies concerning to the initial position is considered.

Let $\{t_0,x_0\}$ be an initial position of system (1.1), where $x_0=x(t_0)$, $\Delta_r$ is the splitting of half-interval $t_0\le t<\infty$, $\tau_1^{(r)},\tau_2^{(r)},\dots$ are splitting nodes, then the diameter of splitting will be $\delta_r=\sup\limits_s(\tau_{s+1}^{(r)}-\tau_s^{(r)})$. It is supposed that at any splitting $\Delta_r$ the instants $\vartheta_j$ ($j=1,\dots,m$) are splitting nodes, i.e.

$$\tau_{s_0}^{(r)}=t_0=\vartheta_0,\ \tau_{s_1}^{(r)}=\vartheta_1,\ \dots,\ \tau_{s_m}^{(r)}=\vartheta_m=\theta. \tag{1.2}$$

The sectionally positioning control $u_i^{(s)}[\tau_s^{(r)},x[\tau_s^{(r)}]]$ and strategies of players $U_i$, the Euler's broken lines

$$x_\Delta^{(s)}\left[\cdot,\tau_s^{(r)},x_\Delta^{(s)}[\tau_s^{(r)}],u_{i_1}[\tau_s^{(r)},x_\Delta^{(s)}[\tau_s^{(r)}]],\dots,u_{i_l}[\tau_s^{(r)},x_\Delta^{(s)}[\tau_s^{(r)}]],u_{i_{l+1}}[t],\dots,u_{i_k}[t]\right],$$

generated by the controls of $l$ players ($1\le l\le k$), are defined on $\left[\tau_s^{(r)},\tau_{s+1}^{(r)}\right)$ ($s=0,1,\dots$) intervals. On the whole interval $[t_0,\theta]$ the motion of system is defined as an absolutely continuous function, for which one can find the Euler's broken lines that uniformly converge to it. By $X[t_0,x_0,U_{i_1},\dots,U_{i_l}]$ we denote the totality of all movements leaving the position $\{t_0,x_0\}$ and generated by the strategies $U_{i_1},\dots,U_{i_l}$ of $l$ players ($1\le l\le k$). It is called the bunch of movements.

**2.** Let the players work for the stability of situation: the choice of strategy for each player is based on some principle, the deviation from which can result in an increase of the gain of other players. Such a principle is the case at the choice of balance set of strategies defined as follows:

*Definition 1.* The set of strategies $\{U_1^o,\dots,U_k^o\}$ is referred to as balanced with respect to the initial position $\{t_0,x_0\}$, if for each number $i\in K$ any movement $x[\cdot]$ from the bunch $X[t_0,x_0,U_1^o,\dots,U_{i-1}^o,U_{i+1}^o,\dots,U_k^o]$ avoids meeting the set $M_i^{(j)}$ up to the instant $\vartheta_j$ for all $j=1,\dots,m$.

The balance set of strategies of players will be constructed using the method of aiming at the corresponding set [1].

Let the set $W$ be given in space $[t_0,\theta]\times R^n$, in which for all $t\in[t_0,\theta]$ $W(t)=\{x\mid\{t,x\}\in W\}\ne\varnothing$ and which satisfies the following two conditions:

*Condition 1.* The set $W$ is closed, and for all $i\in K$ and $j=1,\dots,m$

$$W(\vartheta_j)\cap M_i^{(j)}=\varnothing.$$

*Condition 2.* Irrespective of the position $\{t_*,x_*\}\in W$, the number $i\in K$, the vector $u_i^*(\cdot)\in P_i$ and instant of time $t^*\in[t_*,\theta]$, there are admissible strategies $u_\alpha(\cdot)\in P_\alpha$, $\alpha\in K(i)$, such that for the solution $x(\cdot)$ of differential equation

$$\dot{x}=f(t,x,u_1,\dots,u_{i-1},u_i^*,u_{i+1},\dots,u_k),\quad x(t_*)=x_*, \tag{2.1}$$

it holds that $\{t^*,x(t^*)\}\in W$ (or, what is the same, there is a movement $x(\cdot)$ from a bunch of movements $X[t_*,x_*,U_i^*]$ such that $\{t^*,x(t^*)\}\in W$).

Now define the set of strategies $U_1^e,\dots,U_k^e$ extreme to set $W$, that satisfies the following two conditions [1]:

*Condition 3.* If $\{t,x\}\notin W$ and $W(t)=\{x/\{t,x\}\in W\}\neq\varnothing$, then for all $i\in K$ the strategies $u_i^e=U_i^e(t,x)$ are found from equality

$$\max_{u_\alpha\in P_\alpha(\alpha\in K(i))}(x-w)'f\left(t,x,u_1,\dots,u_{i-1},u_i^e,u_{i+1},\dots,u_k\right)= \\ =\min_{u_i\in P_i}\max_{u_\alpha\in P_\alpha(\alpha\in K(i))}(x-w)'f\left(t,x,u_1,\dots,u_k\right). \tag{2.2}$$

Here $w$ is watever vector (the same for all $i$) satisfying the condition

$$\|x-w\|=\min_{w\in W(t)}\|x-w\|.$$

*Condition 4.* If $\{t,x\}\notin W$, but $W(t)=\varnothing$ or $\{t,x\}\in W$, then $u_i^e(t,x)$ is an arbitrary vector from $P_i$ for all $i\in K$.

***Theorem 1.*** Let the set $W$ satisfies Conditions 1 and 2 and $\{t,x\}\in W$. Then the set of strategies defined by the Conditions 3 and 4 is balanced (in the sense of Definition 1) with respect to the initial position $\{t,x\}$.

*Proof.* Let the $i$-th player select an admissible strategy $U_i$. It will be shown that $x(t^*,t,x,U_1^e,\dots,U_{i-1}^e,U_i,U_{i+1}^e,\dots,U_k^e)\in W(t^*)$ at $t\le t^*\le\theta$.

Now denote as $\left(K(i),\ i,\ W(t),\ M_i^{(j)},\{\vartheta_j\},j=1,\dots,m\right)$ the differential game of two persons, in which the set of players $K(i)$ seeks to hold out the movement in set $W$ by instants $\vartheta_j$, and the $i$-th player seeks to deviate the movement of system from sets $W(\vartheta_j)$, $j=1,\dots,m$.

The strategies defined by Conditions 3 and 4 keep the system state in the set $W$ for any movement from $X\left[t_0,x_0,U_1^e,\dots,U_{i-1}^e,U_{i+1}^e,\dots,U_k^e\right]$ commenced in it up to the instants $\vartheta_j$ ($j=1,\dots,m$). Therefore, the extreme strategies form barriers around the set $W$ that impede the exit of movements $x[t]$ from $W$ up to the instants $\vartheta_j$. Hence, according to [1] (Lemma 15.1), the strategies $U_1^e,\dots,U_{i-1}^e,U_{i+1}^e,\dots,U_k^e$ would hold the system movement on the set $W$ till the instants of time $\vartheta_j$, $j=1,\dots,m$, for any strategy $U_i$, i.e. the set of strategies $U_1^e,\dots,U_{i-1}^e,U_{i+1}^e,\dots,U_k^e$ is balanced initial positions from $W$.

For construction of set $W$ for the following reasoning will be used.

Let the sets $N^{(j)}(j=1,\dots,m)$ are the convex compacts in $R^n$ that satisfy the conditions $N^{(j)}\cap M_i^{(j)}=\varnothing$, $i\in K$, $j=1,\dots,m$.

Let the system reach the position $\{t,x\}$ ($\vartheta_{r-1}\le t<\vartheta_r$). Now define

$$\varepsilon_i^0(t,x) = \max_{\|l_\beta^{(i)}\|=1,\beta=r,\ldots,m} \sum_{j=r}^{m} (< l_j^{(i)}, x > + \min_{-q\in N^{(j)}} < l_j^{(i)}, q > + \\ + \int_t^\theta \max_{u_i\in P_i} \min_{u_\alpha\in P_\alpha(\alpha\in K(i))} < l_j^{(i)}, f^j(\tau, x, u_1, \ldots, u_k) > d\tau), \tag{2.3}$$

if the right hand side is more than zero and $\varepsilon_i^0(t,x) = 0$ otherwise. Here

$$f^j(\tau, x, u_1, ..., u_k) = \begin{cases} f(\tau, x, u_1, ..., u_k) & \text{at } \tau \le \vartheta_j, \\ 0 & \text{at } \tau > \vartheta_j. \end{cases}$$

Let

$$\varepsilon^0(t,x) = \max_{i\in K} \varepsilon_i^0(t,x), \tag{2.4}$$

$\varepsilon_i^0(t,x)$, $\varepsilon^0(t,x)$ being continuous functions of their arguments. Now introduce the following notations:

- denote as $I^{(0)}(t,x)$ the set of maximizing indexes in (2.4) for position $\{t,x\}$, where $\varepsilon^0(t,x) > 0$ and $I^{(0)}(t,x) = K$, if $\varepsilon^0(t,x) = 0$;
- denote as $L_i^0(t,x)$ the totality of sets of vectors $l_j^{(i)}$ $(j = r,...,m)$ maximizing the functions $\varepsilon_i^0(t,x)$ (2.3), if $\varepsilon_i^0(t,x) > 0$ and $L_i^0(t,x)$ completely coincides with the unit sphere in position $\{t,x\}$, where $\varepsilon_i^0(t,x) = 0$;

$$L^0(t,x) = \bigcup_{i\in I^0(t,x)} L_i^0(t,x);$$

$$S_i^0(t,x) = \left\{ s_i = \sum_{j=r}^{m} l_j^{(i)}, l_j^{(i)} \in L_i^0(t,x) \right\}; \quad S^0(t,x) = \bigcup_{i\in I^0(t,x)} S_i^0(t,x).$$

We assume that the following conditions are satisfied:

*Condition 5.* For each number $i \in K$ and for any position $\{t,x\}$, where $\varepsilon_i^0(t,x) > 0$, in (2.3) maxima are reached on a unique set $l_j^{(i)^0}$ $(j = r,...,m)$.

*Condition 6.* In each position $\{t,x\}$, where $\varepsilon^0(t,x) > 0$, for any numbers $i \in I^0\{t,x\}$, vector $s_i \in S_i^0$ and index $\alpha \in K$ there takes place

$$\max_{u_i\in P_i} \min_{u_\alpha\in P_\alpha} < s_i, f(t,x,u_1,...,u_k) > \ \ge \ \max_{u_\alpha\in P_\alpha} \min_{u_i\in P_i} < s_i, f(t,x,u_1,\ldots,u_k) >. \tag{2.5}$$

*Condition 7.* In each position $\{t,x\}$, where $\varepsilon^0(t,x) > 0$, for any number $i \in K$ there is a vector $u_i^0 \in P_i$, such that for all $s_i \in S_i^0$ there takes place an equality

$$\min_{u_i\in P_i} \max_{u_\alpha\in P_\alpha(\alpha\in K(i))} < s, f(t,x,u_1,...,u_k) > = \max_{u_\alpha\in P_\alpha(\alpha\in K(i))} < s, f(t,x,u_1,\ldots,u_{i-1},u_i^0 u_{i+1},...,u_k) >.$$

***Theorem 2.*** Under Conditions 5, 6, 7 the set $W = \{\{t,x\}, \varepsilon^0(t,x) \le 0\}$ will satisfy Conditions 1, 2.

*Proof.* As from the definition of $\varepsilon^0(t,x)$, i.e. from (2.3) and (2.4), there follows the closure of set $W$, hence, the Condition 1 is satisfied. The fulfillment of Condition 2 will be shown by contradiction assuming that there is a position

$\{t_*, x_*\} \in W$, a number $i \in K$, a vector $u_i^0 \in P_i$ and an instant $t^* \in (t_*, \theta)$ such that the solution of differential equation

$$\dot{x} = f(t, x, u_1, ..., u_{i-1}, u_i^0, u_{i+1}, ..., u_k), \quad x(t_*) = x_*, \tag{2.6}$$

for any $u_\alpha$ $(\alpha \in K(i))$, leaves the set $W$ at the instant $t^*$. Now choose vectors $u_\alpha^*$, $\alpha \in K(i)$, satisfying the Condition 7:

$$\begin{aligned} &\max_{u_\beta \in P_\beta (\beta \in K(\alpha))} < s, f(t, x, u_1, ..., u_{\alpha-1}, u_\alpha^*, u_{\alpha+1}, ..., u_k) > = \\ &= \min_{u_\alpha \in P_\alpha} \max_{u_\beta \in P_\beta (\beta \in K(\alpha))} < s, f(t, x, u_1, ..., u_{\alpha-1}, u_\alpha, u_{\alpha+1}, ..., u_k) >, \ s \in S^0(t, x). \end{aligned} \tag{2.7}$$

Here $x(\cdot)$ is a solution of equation (2.6) for controls from (2.7). Then according to above assumptions there is an interval $[t_1, t_2] \subset [t_*, t^*]$ such that $\varepsilon^0(t, x) > 0$ almost for all $t \in [t_1, t_2]$ and $\varepsilon^0(t_1, x(t_1)) < \varepsilon^0(t_2, x(t_2))$.

From [1, 2] it follows that $\varepsilon^0 : t \to \varepsilon^0(t, x(t))$ is a differentiable function of $t$ almost everywhere on $[t_1, t_2]$ and $\exists\, p \in I^0(t, x(t))$

$$\begin{aligned} \frac{d\varepsilon^0(t, x(t))}{dt} = \frac{d\varepsilon_p(t, x(t))}{dt} = \sum_{j=r}^{m} < l_j^{(p)}, f(t, x, u_1^*, ..., u_{i-1}^*, u_i^0, u_{i-1}^*, ..., u_k^*) > - \\ - \sum_{j=r}^{m} \min_{u_p \in P_p} \max_{u_\alpha \in P_\alpha (\alpha \in K(p))} < l_j^{(p)} f(t, x, u_1, ..., u_{p-1}, u_p, u_{p+1}, \ldots, u_k) >. \end{aligned} \tag{2.8}$$

After transformations with due regard for (2.7) from (2.8) we obtain

$$\begin{aligned} \frac{d\varepsilon(t, x(t))}{dt} \le \max_{u_i \in P_i} \min_{u_\rho \in P_\rho} \min_{\substack{u_\alpha \in P_\alpha, \\ \alpha \in K, \alpha \ne p, i}} < s^\rho, f(t, x, u_1, ..., u_k) > - \\ - \max_{u_p \in P_p} \min_{u_i \in P_i} \min_{\substack{u_\alpha \in P_\alpha, \\ \alpha \in K, \alpha \ne p, i}} < s^\rho, f(t, x, u_1, ..., u_k) > \le 0. \end{aligned}$$

The validity of the last inequality follows from Condition 6. It turns out that almost everywhere on the interval $[t_1, t_2]$ $\frac{d\varepsilon_p}{dt} \le 0$, but $\varepsilon^0(t_1, x(t_1)) \ge \varepsilon^0(t_2, x(t_2))$, what contradicts the assumption. Hence, the Condition 2 holds.

Thus, it is shown that for differential several person games in case of many aim sets the strategies, extreme to a corresponding stable set, make a balance set with respect to initial positions.

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