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Mathematics

ON THE STRUCTURE OF WIENER–HOPF OPERATOR CORRESPONDING TO ANISOTROPIC BOUNDARY VALUE PROBLEM, CONNECTED WITH HELMHOLTZ–SCHRÖDINGER EQUATION WITH THE BOUNDARY CONDITIONS OF THE FIRST AND SECOND TYPE

S. A. HOSSEINY MATIKOLAI*

Chair of Differential Equations and Functional Analysis, YSU

In this paper we investigate the Fredholm property of Wiener–Hopf operator for anisotropic boundary value problem for the Helmholtz–Schrödinger equation with the boundary conditions of the first and second type on the line $y = 0$.

***Keywords*:** Wiener–Hopf operator, Fredholm property.

First we formulate the anisotropic problem (AP)

Let $\Omega^{\pm} = \{(x, y) \in R^2 : y > 0 (y < 0)\}$. It is necessary to find the function $u \in L^2(R^2)$ such that $u|_{\Omega^{\pm}} \in H^1(\Omega^{\pm})$ and the following equation

$$\begin{cases} \Delta u + (k_+^2 + 2\beta_+^2 \sec h^2(\beta_+ y))u = 0 & \text{in } \Omega^+, \\ \Delta u + (k_-^2 + 2\beta_-^2 \sec h^2(\beta_- y))u = 0 & \text{in } \Omega^-, \end{cases} \tag{1}$$

where $\operatorname{Im} k_{\pm} > 0$ and boundary conditions are satisfied

$$\begin{cases} \begin{cases} a_0^{(+)} u(x_j + 0) + b_0^{(+)} u(x_j - 0) = h_0^{(+)}(x) \\ a_1^{(+)} \dfrac{\partial u(x_j + 0)}{\partial y} + b_1^{(+)} \dfrac{\partial u(x_j - 0)}{\partial y} = h_1^{(+)}(x) \end{cases} & \text{on } R^+, \\ \begin{cases} a_0^{(-)} u(x_j + 0) + b_0^{(-)} u(x_j - 0) = h_0^{(-)}(x) \\ a_1^{(-)} \dfrac{\partial u(x_j + 0)}{\partial y} + b_1^{(-)} \dfrac{\partial u(x_j - 0)}{\partial y} = h_1^{(-)}(x) \end{cases} & \text{on } R^+. \end{cases}$$

Here $h_0^{(+)} \in H^{1/2}(R^+), h_1^{(+)} \in H^{-1/2}(R^+)$, $h_0^{(-)} \in H^{1/2}(R^-)$, $h_1^{(-)} \in H^{-1/2}(R^-)$, and $a_i^{(\pm)}, b_i^{(\pm)}\ (i = 0,1)$ are complex constants. Note that $H^1(\Omega^{\pm})$ and $H^{\pm 1/2}(\Omega^{\pm})$ are the corresponding Sobolev spaces (see [1]).

Let $\chi_{\pm}$ are the characteristic functions respectively for $R^{\pm}$ and $\hat{u}(\lambda, \mu)$ is Fourier transform of $u(x, y)$ function:

* E-mail: seyedalim895@yahoo.com

$$\hat{u}(\lambda,\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\lambda x+\mu y)}u(x,y)dxdy,\ \gamma_{\pm}(\lambda)=\sqrt{\lambda^2-k_{\pm}^2}$$

with slitted branches $\pm k_{\pm}\pm i\tau,\tau>0$, and $\gamma_{\pm}(\lambda)\sim\lambda$ by $|\lambda|\to\infty$.

The solution of (1) is

$$u(x,y)=G\begin{pmatrix}u_0^{(+)}\\ u_0^{(-)}\end{pmatrix}(x,y)=F_{\lambda\mapsto x}^{-1}\left\{e^{-y\gamma_+(\lambda)}\hat{u}_0^{(+)}(\lambda)\chi_+(y)+e^{y\gamma_-(\lambda)}\hat{u}_0^{(-)}(\lambda)\chi_-(y)\right\}$$

(see [2–4], where $u_0^{(\pm)}=u(x,\pm 0)$.

As in [2] we introduce the operators

$$B_{\pm}=F^{-1}L_{\pm}F.$$

Here the matrix-functions $L_{\pm}(\lambda)$ have the form

$$L_{\pm}(\lambda)=\begin{pmatrix}a_0^{(\pm)} & b_0^{(\pm)}\\ -a_1^{(\pm)}\dfrac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)} & b_1^{(\pm)}\dfrac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)}\end{pmatrix}. \qquad (2)$$

Now we shall verify that $B_{\pm}:H^{1/2}(R)^2\to H^{1/2}(R)\times H^{-1/2}(R)$. If $\vec{u}\in H^{1/2}(R)^2$, then $\vec{u}=\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\end{pmatrix}\in F(H^{1/2}(R)^2)\cong H^{1/2}(R)^2$ and

$$L_{\pm}(\lambda)\hat{\vec{u}}=\begin{pmatrix}a_0^{(\pm)} & b_0^{(\pm)}\\ -a_1^{(\pm)}\dfrac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)} & b_1^{(\pm)}\dfrac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)}\end{pmatrix}\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\end{pmatrix}=$$

$$=\begin{pmatrix}a_0^{(\pm)}\hat{u}_1+b_0^{(\pm)}\hat{u}_2\\ -a_1^{(\pm)}\dfrac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)}\hat{u}_1+b_1^{(\pm)}\dfrac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)}\hat{u}_2\end{pmatrix}=\begin{pmatrix}w_1(\lambda)\\ w_2(\lambda)\end{pmatrix}.$$

From here we get

$$\|L_{\pm}(\lambda)\|=\sqrt{\left|a_0^{(\pm)}+b_0^{(\pm)}\right|^2+\left|-a_1^{(\pm)}\frac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)}+b_1^{(\pm)}\frac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)}\right|^2}=O(|\lambda|),$$

as $|\lambda|\to\infty$, therefore, $w_1\in H^{1/2}(R), w_2\in H^{1/2}(R)$. As a result, we obtain $F^{-1}(L_{\pm}(\lambda)\hat{\vec{u}})\in H^{1/2}(R)\times H^{-1/2}(R)$. Thus, $B_{\pm}:H^{1/2}(R)^2\to H^{1/2}(R)\times H^{-1/2}(R)$. From representation (2) it immediately follows that

$$(\det L_{\pm}(\lambda))^{\pm 1}=O(|\lambda|^{\pm 1}) \text{ by } |\lambda|\to\infty. \qquad (3)$$

(3) is equivalent to the bijectivity of $B_{\pm}$ operators. We also assume that the following conditions are fulfilled

$$\begin{cases}a_0^{(\pm)}b_1^{(\pm)}\dfrac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)}+a_1^{(\pm)}b_0^{(\pm)}\dfrac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)}\neq 0, & \lambda\in R,\\ a_0^{(\pm)}b_1^{(\pm)}+a_1^{(\pm)}b_0^{(\pm)}\neq 0.\end{cases}$$

Consequently, the problem AP is a normal type problem and in virtue of the theorem on equivalence (see [2, 3]) the function $u\in L^2(R^2)$ is the solution of this problem, only if

a) u is represented in the form (3) and $u_0^{(\pm)}$ satisfy the system

$$\begin{pmatrix} u_0^{(+)} \\ u_0^{(-)} \end{pmatrix} = B_-^{-1}\left\{\begin{pmatrix} v_+ \\ w_+ \end{pmatrix} + \begin{pmatrix} l_{(r)}h_0^{(-)} \\ l_{(nr)}h_1^{(-)} \end{pmatrix}\right\},$$

where $l_{(r)}, l_{(nr)}$ are even and odd continuations on the whole axis,

b) the functions v_+, w_+ are solutions of the Wiener–Hopf system

$$W\begin{pmatrix} v_+ \\ w_+ \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} - \chi_+ B_+ B_-^{-1}\begin{pmatrix} l_{(r)}h_0^{(-)} \\ l_{(nr)}h_1^{(-)} \end{pmatrix} = \begin{pmatrix} h_0^* \\ h_1^* \end{pmatrix}. \tag{4}$$

Here, as above, it is easy to check that

$$W = \chi_+ F^{-1} L F : \tilde{H}^{1/2}(R^+)\times\tilde{H}^{-1/2}(R^+) \to H^{1/2}(R^+)\times H^{-1/2}(R^+),$$

where

$$L(\lambda) = L_+(\lambda)L_-^{-1}(\lambda). \tag{5}$$

To prove (5) note that $L_\pm(\lambda)$ matrices are invertible and

$$L_-^{-1}(\lambda) = \frac{1}{a_0^{(-)}b_1^{(-)}\dfrac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)} + a_1^{(-)}b_0^{(-)}\dfrac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)}}\begin{bmatrix} b_1^{(-)}\dfrac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)} & -b_0^{(-)} \\ a_1^{(-)}\dfrac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)} & a_0^{(-)} \end{bmatrix},$$

consequently we can write

$$L_+(\lambda)L_-^{-1}(\lambda) = \frac{1}{a_0^{(-)}b_1^{(-)}\dfrac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)} + a_1^{(-)}b_0^{(-)}\dfrac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)}}\times$$

$$\times\begin{bmatrix} a_0^{(+)}b_1^{(-)}\dfrac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)} + a_1^{(-)}b_0^{(+)}\dfrac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)} & a_0^{(-)}b_0^{(+)} - a_0^{(+)}b_0^{(-)} \\ \left(a_1^{(-)}b_1^{(+)} - a_1^{(+)}b_1^{(-)}\right)\dfrac{(\gamma_+^2(\lambda)-\beta_+^2)(\gamma_-^2(\lambda)-\beta_-^2)}{\gamma_+(\lambda)\gamma_-(\lambda)} & a_0^{(-)}b_1^{(+)}\dfrac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)} + a_1^{(+)}b_0^{(-)}\dfrac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)} \end{bmatrix} = L(\lambda).$$

Theorem. Let the problem AP is of normal type. Then the operator W given by (4) has the Fredholm property, iff the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{a_0^{(-)}b_1^{(-)} + a_1^{(-)}b_0^{(-)}}\begin{bmatrix} a_0^{(+)}b_1^{(-)} + a_1^{(-)}b_0^{(+)} & a_0^{(-)}b_0^{(+)} - a_0^{(+)}b_0^{(-)} \\ a_1^{(-)}b_1^{(+)} - a_1^{(+)}b_1^{(-)} & a_1^{(+)}b_0^{(-)} + a_0^{(-)}b_1^{(-)} \end{bmatrix}$$

satisfies the conditions

$$ad = 0, \quad bc \neq 0 \tag{6}$$

or

$$ad \neq 0, \quad \frac{1}{d} = \frac{bc}{ad} \notin [0,1]. \tag{7}$$

In both cases the index of the operator W is equal to zero.

(The parameter $\lambda = \dfrac{ad}{bc}$ is called the characteristic parameter of the problem AP).

Proof. Due to (5), the symbol of $L(\lambda)$ can be calculated.

Let $\gamma(\lambda) = \sqrt{\lambda^2 - k^2}$ is an auxiliary "wave" corresponding to the number k (we assume that $k_+ = k_-$) and $\gamma(\lambda) = t_-(\lambda) t_+(\lambda) = \sqrt{\lambda - k}\sqrt{\lambda + k}$, where $t_-(\lambda)$, $t_+(\lambda)$ are the branches selected in the usual manner as noted above.

It is well known (see [1]) that the transforms

$$F^{-1} t_+^r F : \tilde{H}^s(R^+) \to \tilde{H}^{s-\frac{r}{2}}(R^+),\ \chi_+ F^{-1} t_-^r F l : H^s(R^+) \to H^{s-\frac{r}{2}}(R^+)$$

are bijective for every continuation l and $s, r \in R$. The operator W is equivalent (coincided modulo invertible operators) to the operator

$$W_0 = \chi_+ F^{-1} L_0 F : L^2(R^+)^2 \to L^2(R^+)^2,$$

where the matrix-function $L_0(\lambda)$ is given as

$$L_0(\lambda) = \begin{pmatrix} t_-(\lambda) & 0 \\ 0 & \dfrac{1}{t_-(\lambda)} \end{pmatrix} L(\lambda) \begin{pmatrix} t_+(\lambda) & 0 \\ 0 & \dfrac{1}{t+(\lambda)} \end{pmatrix} = \frac{1}{a_0^{(-)} b_1^{(-)} \dfrac{\gamma_-^2(\lambda) - \beta_-^2}{\gamma_-(\lambda)} + a_1^{(-)} b_0^{(-)} \dfrac{\gamma_+^2(\lambda) - \beta_+^2}{\gamma_+(\lambda)}} \times$$

$$\times \begin{bmatrix} \left(a_0^{(+)} b_1^{(-)} \dfrac{\gamma_-^2(\lambda) - \beta_-^2}{\gamma_-(\lambda)} + a_1^{(-)} b_0^{(+)} \dfrac{\gamma_+^2(\lambda) - \beta_+^2}{\gamma_+(\lambda)} \right) \dfrac{t_-(\lambda)}{t_+(\lambda)} & a_0^{(-)} b_0^{(+)} - a_0^{(+)} b_0^{(-)} \\ \left(a_1^{(-)} b_1^{(+)} - a_1^{(+)} b_1^{(-)} \right) \dfrac{(\gamma_-^2(\lambda) - \beta_-^2)(\gamma_+^2(\lambda) - \beta_+^2)}{\gamma_-(\lambda)\gamma_+(\lambda)\gamma(\lambda)} & \left(a_0^{(-)} b_1^{(+)} \dfrac{\gamma_-^2(\lambda) - \beta_-^2}{\gamma_-(\lambda)} + a_1^{(+)} b_0^{(-)} \dfrac{\gamma_+^2(\lambda) - \beta_+^2}{\gamma_+(\lambda)} \right) \dfrac{t_+(\lambda)}{t_-(\lambda)} \end{bmatrix}.$$

This matrix-function $L_0(\lambda)$ is invertible in $C(R)^{2\times 2}$, and in case of $|\lambda| \to \infty$ tends to the matrix

$$L_0(\pm\infty) = \begin{pmatrix} \pm a & b \\ c & \pm d \end{pmatrix},$$

i.e. $L_0(\lambda)$ is piecewise continuous on the one-point compactification of $\dot{R}^1$.

In virtue of the theory of Cauchy integral equations on the circle that is obtained by means of the Cayley transform of the real axis (see [5]), the Fredholm property of the operator W_0 (and hence W) is equivalent to the conditions

$$\begin{cases} \det L_0(\lambda) \neq 0, & \lambda \in R, \\ \det[\mu L_0(-\infty) + (1-\mu) L_0(+\infty)] \neq 0, & \mu \in [0,1]. \end{cases}$$

Note that these conditions can be written in the form of the conditions (6), (7). Using the formulas for the index we obtain

$$\operatorname{Ind} W = \dim \operatorname{Ker}(W) - \dim \operatorname{Ker} R(W) = \operatorname{Ind} W_0 = -\operatorname{Ind} L_0(\lambda),$$

where

$$\det L_0(\lambda) = \det L(\lambda) = \frac{\det L_+(\lambda)}{\det L_-(\lambda)} = \frac{a_0^{(+)} b_1^{(+)} \dfrac{\gamma_-^2(\lambda) - \beta_-^2}{\gamma_-(\lambda)} + a_1^{(+)} b_0^{(+)} \dfrac{\gamma_+^2(\lambda) - \beta_+^2}{\gamma_+(\lambda)}}{a_0^{(-)} b_1^{(-)} \dfrac{\gamma_-^2(\lambda) - \beta_-^2}{\gamma_-(\lambda)} + a_1^{(-)} b_0^{(+)} \dfrac{\gamma_+^2(\lambda) - \beta_+^2}{\gamma_+(\lambda)}}.$$

It is easy to see that $\operatorname{Ind} L_0(\lambda) = 0$.

The proof is complete.

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