PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2011, № 2, p. 63–66

COMMUNICATIONS

Mathematics

THE GENERALIZED ENTROPIC PROPERTY FOR THE MONO-*n*-ARY ALGEBRA

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In this paper we show by using the laws of pseudo-distributivity that every idempotent and commutative algebra with one *n*-ary operation satisfying the generalized entropic property is entropic.

***Keywords*:** complex algebra, mode, entropic algebra, generalized entropic property.

Introduction. Given an algebra $A=(A,F)$ we define complex operations for every $\phi \neq A_1,...,A_n \subseteq A$ and every *n*-ary $f \in F$ on the set $\rho(A)$ of all non-empty subsets of the set *A* by $f(A_1,...,A_n)=\{f(a_1,...,a_n): a_i \in A_i\}$. The algebra $Cm\,A=(\rho(A),F)$ is called the complex algebra of *A*.

The complex algebras (called also the globals or the powers of algebras) were studied by several authors [1–7].

The notion of complex operations is widely used. In groups, for instance, a coset *xN* is the complex product of the singleton $\{x\}$ and the subgroup *N*. For a lattice *L,* the set Id *L* of its ideals forms a lattice under the set inclusion. If *L* is distributive, then its joint and meet in Id *L* are precisely the complex operations obtained from the joint and meet of *L,* so Id *L* is a subalgebra of *Cm L.*

Now, consider the set *CSub A* of all (non-empty) subalgebras of algebra *A.* This set may or may not be closed under complex operations. For instance, if A is an Abelian group, it is closed; however, for the majority of groups it is not closed. In the former case, *CSub A* is a subuniverse of *Cm A* and we call it a complex algebra of subalgebras. We will say that *A* has the complex algebra of subalgebras or that *CSub A* exists.

The algebra $A=(A,F)$ is called entropic (or medial), if it satisfies the identity of mediality:

$$g(f(x_{11},...,x_{n1}),...,f(x_{1m},...,x_{nm}))=f(g(x_{11},...,x_{1m}),...,g(x_{n1},...,x_{nm})) \quad (1)$$

for every *n*-ary $f \in F$ and *m*-ary $g \in F$.

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In other words, the algebra A is medial, if it satisfies the hyperidentity of mediality [8–10]. Note that a groupoid is entropic, iff it satisfies the identity of mediality [11]: $xy.uv \approx xu.yv$.

Following [12], the algebra $A=(A,f)$ with one n-ary operation is called a mono-n-ary algebra. It could be entropic, iff it satisfies the identity:

$$f(f(x_{11},...,x_{n1}),...,f(x_{1n},...,x_{nn})) \approx f(f(x_{11},...,x_{1n}),..,f(x_{n1},...,x_{nn})).$$

A variety V is called entropic (or medial), if every algebra in V is entropic. Algebra A is called idempotent (commutative), if every operation of A is idempotent (commutative). An n-ary operation f is called commutative, if $f(x_1,x_2,...,x_n)=f(x_{\alpha(1)},x_{\alpha(2)},...,x_{\alpha(n)})$, where $\alpha \in S_n$. The n-ary operation f is called idempotent, if $f(x,...,x)=x$. An idempotent entropic algebra is called a mode [6].

Definition 1. We say that a variety V (respectively, the algebra A) satisfies the generalized entropic property, if for every n-ary operation f and m-ary operation g of V (of A) there exist m-ary terms $t_1,...,t_n$ such that in V (in A) the below identity holds

$$g(f(x_{11},...,x_{n1}),...,f(x_{1m},...,x_{nm})) = f(t_1(x_{11},...,x_{1m}),...,t_n(x_{n1},...,x_{nm})). \quad (2)$$

For example, a groupoid satisfies the generalized entropic property, if there are binary terms t and s such that $xy.uv \approx t(x,u).s(y,v)$.

It was proved in [13] that for the variety V of groupoids every groupoid in V has the complex algebra of subalgebras, iff V satisfies the above identity for some t and s.

Example 1. Let R be a ring with a unit, G be a subgroup of the multiplicative monoid of R, and X be a subset of G closed under conjugation by elements of X and closed under the mapping $x \mapsto 1-x$, where – is the ring subtraction.

If M is a left module over the ring R, we define for every element $r \in R$ a binary operation $\underline{r}: M^2 \to M$ by : $\underline{r}(x,y)=(1-r)x+ry$.

Of course, the groupoid $(M, \underline{r})$ is idempotent and entropic for every $r \in R$. Now consider the algebra $\underline{M}=(M, \underline{X})$, where $\underline{X}=\{\underline{r} | r \in R\}$. For every $r,t \in X$ we put $s_1=(1-r)^{-1}t(1-r) \in X$ and $s_2=r^{-1}tr \in X$, and we get

$$\underline{t}(\underline{r}(x_1,x_2),\underline{r}(y_1,y_2)) \approx (1-t)(1-r)x_1+(1-t)rx_2+t(1-r)y_1+try_2 \approx$$
$$(1-r)(1-s_1)x_1+r(1-s_2)x_2+(1-r)s_1y_1+rs_2y_2 \approx \underline{r}(\underline{s_1}(x_1,y_1),\underline{s_2}(x_2,y_2)).$$

So, the algebra $\underline{M}$ satisfies the generalized entropic property. On the other hand, it is entropic, iff $rt=tr$ for all $r,t \in X$. To check this we put $x_1=y_1=y_2=0$ and $x_2=1$ in the previous identity.

If R is a non-commutative division ring, G is its multiplicative group and $X=R \setminus \{0,1\}$, then $\underline{M}$ is a non-entropic idempotent algebra satisfying the generalized entropic property.

Theorem 1. Every algebra in a variety V has the complex algebra of subalgebras, iff the variety V satisfies the generalized entropic property.

Proof. In [7].

The Main Result. The mono-n-ary algebra $A=(A,f)$ satisfies the generalized entropic property, if there are n-ary terms $t_1,...,t_n$ such that the below identity holds:

$$f(f(x_{11},...,x_{n1}),...,f(x_{1n},...,x_{nn}))\approx f(t_1(x_{11},...,x_{1n}),..,t_n(x_{n1},...,x_{nn})).$$

Immediate consequences of the generalized entropic property in the idempotent algebra $A=(A,f)$ with one n-ary operation are the following identities that can be treated as the laws of pseudo-distributivity:

$$f(t_1(x_{11},...,x_{1n}),x_{21},...,x_{n1})\approx f(f(x_{11},x_{21},...,x_{n1}),f(x_{12},x_{21},...,x_{n1}),...,f(x_{1n},x_{2n},...,x_{n1})),$$
$$\vdots$$
$$f(x_{11},...,x_{(n-1)1},t_n(x_{n1},...,x_{nn}))\approx f(f(x_{11},...,x_{(n-1)1},x_{n1}),...,f(x_{11},...,x_{(n-1)1},x_{nn})).$$

Theorem 2. An idempotent and commutative mono-n-ary algebra $A=(A,f)$ satisfying the generalized entropic property is entropic.

Proof. Using the pseudo-distributivity and the commutativity we obtain

$$f(t_1(x_{11},...,x_{1n}),x_{21},...,x_{n1})\approx$$
$$f(f(x_{11},x_{21},...,x_{n1}),f(x_{12},x_{21},...,x_{n1}),...,f(x_{1n},x_{2n},...,x_{n1}))\approx$$
$$f(f(x_{21},...,x_{n1},x_{11}),f(x_{21},...,x_{n1},x_{12}),...,f(x_{2n},...,x_{n1},x_{1n}))\approx$$
$$f(x_{21},...,x_{n1},t_n(x_{11},...,x_{1n}))\approx f(t_n(x_{11},...,x_{1n}),x_{21},...,x_{n1}).$$

Thus, we can show by the same manner that

$$f(t_1(x_{11},...,x_{1n}),x_{21},...,x_{n1})\approx$$
$$f(x_{n1},t_2(x_{11},...,x_{1n}),x_{21},...,x_{(n-1)1})\approx...\approx f(x_{21},...,x_{n1},t_n(x_{11},...,x_{1n})).$$

Now using the property of idempotency and the above identity we have

$$t_1(x_{11},...,x_{1n})\approx f(t_1(x_{11},...,x_{1n}),...,t_1(x_{11},...,x_{1n}))\approx$$
$$f(t_1(x_{11},...,x_{1n}),t_2(x_{11},...,x_{1n}),t_1(x_{11},...,x_{1n}),...,t_1(x_{11},...,x_{1n}))\approx...\approx$$
$$f(t_1(x_{11},...,x_{1n}),t_2(x_{11},...,x_{1n}),...,t_n(x_{11},...,x_{1n}))\approx f(x_{11},...,x_{1n}).$$

Similarly, for $t_2,...,t_n$ we have

$$t_2(x_{21},...,x_{2n})\approx f(x_{21},...,x_{2n}),$$
$$\vdots$$
$$t_n(x_{n1},...,x_{nn})\approx f(x_{n1},...,x_{nn}).$$

Thus, from the generalized entropic property and the last identities we have

$$f(f(x_{11},...,x_{n1}),...,f(x_{1n},...,x_{nn}))\approx f(t_1(x_{11},...,x_{1n}),..,t_n(x_{n1},...,x_{nn}))\approx$$
$$f(f(x_{11},...,x_{1n}),...,f(x_{n1},...,x_{nn})).$$ □

Received 09.12.2010

REFERENCES

1. **Grätzer G., Lakser H.** Colloq. Math., 1988, № 56, p. 19–29.
2. **Brink C.** Algebra Universe, 1993, № 30, p. 177–216.

3. **Bošnjak I. and Madarász R.** Algebra and Discr. Math., 2003, № 2, p. 14–35.
4. **Romanowska A., Smith J.D.H.** Geometry and Convexity. Berlin: Heldermann Verlag, 1985.
5. **Romanowska A., Smith J.D.H.** J. Algebra, 1989, № 12, p. 263–283.
6. **Romanowska A., Smith J.D.H.** Modes. World Scientific, 2002.
7. **Adaricheva K., Pilitowska A., Stanovský D.** On Complex Algebras of Subalgebras. arXiv:math/0610832v1 [math.RA] 2006.
8. **Movsisyan Yu.M.** Introduction to the Theory of Algebras with Hyperidentities. Yer.: YSU Press, 1986 (in Russian).
9. **Movsisyan Yu.M.** Hyperidentities and Hypervarieties in Algebras, Yer.: YSU Press, 1990 (in Russian).
10. **Movsisyan Yu.M.** Math. Surveys, 1998, № 53, p. 57–108 (in Russian).
11. **Ježek J., Kepka T.** Medial Groupoids. Rozpravy ČSAV, 1983, № 93/2.
12. **Burris S., Sankappanavar H. P.** A Course in Universal Algebra. Springer-Verlag, 1981.
13. **Evans T.** J. London Math. Soc., 1962, № 35, p. 53–59.