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Mathematics

CHORD LENGTH DISTRIBUTION FUNCTION FOR LENS

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In the paper a formula for the chord length distribution function of a lens is obtained. In the special case of a regular lens the table of values of the chord length distribution function is given.

***Keywords*:** chord length distribution function, regular lens.

1. Introduction. Let G be the space of all lines g in the Euclidean plane R^2, (p,φ) are the polar coordinates of the foot of the perpendicular to g from the origin O, be standard coordinates for line $g\in G$. Let $\mu(\cdot)$ stand for locally finite measure on G invariant with respect to the group M of all Euclidean motions (translations and rotations). It is well-known that the element of the measure up to a constant factor (see [1, 2]) has the following form $\mu(dg)=dg=dpd\varphi$, where dp is the one dimensional Lebesgue measure, while $d\varphi$ is the normalised measure on the unit circle.

For bounded convex domain D denote by $[D]=\{g\in G: g\cap D\neq\phi\}$ the set of lines that intersect D. Then (see [1, 3]) $\mu([D])=|\partial D|$, where ∂D is the boundary of D and $|\partial D|$ stands for the length of ∂D.

A random line in $[D]$ is one with distribution proportional to restriction of μ to $[D]$, so,

$$P(A)=\frac{\mu(A)}{|\partial D|} \quad \text{for any Borel } A\subset[D]. \tag{1}$$

Furthermore, let A_D^y be the set of lines that intersect D and produce a chord $\chi(g)=g\cap D$ of length less than y: $A_D^z=\{g\in[D]:|\chi(g)|\le z\},\quad z\in R$.

Distribution function of the length of a random chord χ of D is defined as

$$F(z)=\frac{1}{|\partial D|}\mu(A_D^z)=\frac{1}{|\partial D|}\iint_{A_D^z} d\phi\, dp\,. \tag{2}$$

Therefore, to obtain chord length distribution function for a bounded convex domain D we have to calculate the integral in the right-hand side of (2). Explicit formulae for the chord length distribution functions are known only for the cases of a disc, a rectangle and a regular n-gon (see [4–7]).

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The main result of the paper is an expression for the chord length distribution function for a lens.

2. The Case of a General Lens. Consider two discs D_1 and D_2 with centers $(0,-a)$ and $(0,b)$ ($a>0, b>0$) and radii r and R ($r \le R$) respectively. We consider a case, when the intersection D of these discs is not empty. D is called a lens. We denote the boundary of D by $\Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are arcs of the circles ∂D_1 and ∂D_2 respectively. The lengths of Γ_1 and Γ_2 are equal to $2r\arcsin\frac{a}{r}$ and $2R\arcsin\frac{b}{R}$ respectively. It follows from (2) that we have to calculate the integral on the right-hand side of (2). For any lens the minimal chord length is 0, and maximal length is $2\sqrt{r^2-a^2}$ ($\sqrt{r^2-a^2}=\sqrt{R^2-b^2}$).

For $0<z\le 2\sqrt{r^2-a^2}$ we split A_D^z into 3 parts by the following way:

$$A_D^z = A_1^z \cup A_2^z \cup A_3^z,$$

$$A_1^z = \{g\in[D]: |\chi(g)|\le z \text{ and } P_1\in\Gamma_1,\ P_2\in\Gamma_1\},$$

$$A_2^z = \{g\in[D]: |\chi(g)|\le z \text{ and } P_1\in\Gamma_2,\ P_2\in\Gamma_2\},$$

$$A_3^z = \{g\in[D]: |\chi(g)|\le z \text{ and } P_1\in\Gamma_1,\ P_2\in\Gamma_2\},$$

where $|\chi(g)|=|P_1,P_2|$ is the distance between P_1 and P_2.

It is obvious that $A_i^z\cap A_j^z=\varnothing$, if $i\ne j$, so

$$\iint\limits_{A_D^z} d\phi\, dp = \iint\limits_{A_1^z} d\phi\, dp + \iint\limits_{A_2^z} d\phi\, dp + \iint\limits_{A_3^z} d\phi\, dp := I_1+I_2+I_3.$$

We consider the integral I_1. If $P_1,\ P_2\in\Gamma_1$, then $\varphi\in(0,\pi)$. We consider only the case $\varphi\in\left(0,\frac{\pi}{2}\right)$, because for $\varphi\in\left(\frac{\pi}{2},\pi\right)$ the situation is the same. It is easy to verify that the intersection points of $g=(\varphi,p)$ with lens can belong to Γ_1, only if $\varphi\ge\arcsin\frac{a}{r}$. For $\varphi\in\left(\arcsin\frac{a}{r},\frac{\pi}{2}\right)$ let's find the values of p, for which those intersections belong to Γ_1. Note, that if $P_1,\ P_2\in\Gamma_1$ and the p-coordinate of the chord $\chi(\varphi,p)$ belongs to the interval (p_1,p_2), then $p_1=\sqrt{r^2-a^2}\cos\varphi$. Let us find p_2. The line $g=(\varphi,p_2)$ has to be tangent to Γ_1 at a point x_0, and the coefficient of that line is $-\cot\varphi$.

The equation of the curve Γ_1 is $y=\sqrt{r^2-x^2}-a$ or $\begin{cases} x=r\cos\psi, \\ y=r\sin\psi-a, \end{cases}$ where $\psi\in\left(\arcsin\frac{a}{r},\pi-\arcsin\frac{a}{r}\right)$ is the central angle of the circle ∂D_1. So, we have $-\frac{x_0}{\sqrt{r^2-x_0^2}}=-\cot\varphi$, or $x_0=r\cos\varphi$. Comparing with equations of the line $g=(\varphi,p_2)$ (the normal equation) and the tangent, we get $p_2=r\sin\varphi-a$ (we can also prove this easily using elementary geometrical rules). Therefore, $P_1,\ P_2 \in \Gamma_1$, iff $g=(\varphi,p)\in\left(\arcsin\frac{a}{r},\pi-\arcsin\frac{a}{r}\right)\times\left(\sqrt{r^2-a^2}\,|\cos\varphi|,r\sin\varphi-a\right)$, because Γ_1 is symmetric with respect to y-axis. In order to find the coordinates of the intersection points of $g=(\varphi,p)$ with curve Γ_1 we have to solve the system of equations

$$x\cos\varphi+y\sin\varphi-p=0,\quad x=r\cos\psi,\quad y=r\sin\psi-a. \tag{3}$$

From (3) we get $r\cos(\varphi-\psi)=p+a\sin\varphi$, $\psi=\varphi\pm\arccos\frac{p+a\sin\varphi}{r}$. Therefore,

$$\begin{cases} x_1=r\cos\left(\varphi-\arccos\dfrac{p+a\sin\varphi}{r}\right),\\ y_1=r\sin\left(\varphi-\arccos\dfrac{p+a\sin\varphi}{r}\right)-a \end{cases} \quad\text{and}\quad \begin{cases} x_2=r\cos\left(\varphi+\arccos\dfrac{p+a\sin\varphi}{r}\right),\\ y_2=r\sin\left(\varphi+\arccos\dfrac{p+a\sin\varphi}{r}\right)-a. \end{cases}$$

Now we can find the value of p, for which $|\chi(\varphi,p)|=z$.

$$z=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\ .$$

After simplification we obtain

$$z=r\sqrt{2-2\cos\left(2\arccos\frac{p+a\sin\varphi}{r}\right)}=2r\sin\left(\arccos\frac{p+a\sin\varphi}{r}\right)=2r\sqrt{1-\left(\frac{p+a\sin\varphi}{r}\right)^2}.$$

Therefore, we get $p=r\sqrt{1-\frac{z^2}{4r^2}}-a\sin\varphi$. For $\varphi\in\left(\arcsin\frac{a}{r},\frac{\pi}{2}\right)$ the line $g=(\varphi,p)$ for various values of p $\left(p\in\left(\sqrt{r^2-a^2}\cos\varphi,r\sin\varphi-a\right)\right)$ can make chords from interval $\left(0,2r\sin\left(\varphi-\arcsin\frac{a}{r}\right)\right)$. So, if $0\le z\le 2\sqrt{r^2-a^2}$, then

$$\begin{aligned} I_1&=2\left[\int_{\arcsin\frac{a}{r}}^{\arcsin\frac{a}{r}+\arcsin\frac{z}{2r}}d\varphi\int_{\sqrt{r^2-a^2}\cos\varphi}^{r-a\sin\varphi}dp+\int_{\arcsin\frac{a}{r}+\arcsin\frac{z}{2r}}^{\frac{\pi}{2}}d\varphi\int_{r\sqrt{1-\frac{z^2}{4r^2}}-a\sin\varphi}^{r-a\sin\varphi}dp\right]= \\ &=2r\left(\arccos\frac{a}{r}-\frac{z}{2r}+\sqrt{1-\frac{z^2}{4r^2}}\left(\arcsin\frac{a}{r}-\arccos\frac{z}{2r}\right)\right). \end{aligned} \tag{4}$$

To find I_2 we can change a by b and r by R in (4), because measure $d\varphi dp$ is invariant with respect to M. Now we calculate I_3. By symmetry, we calculate integral I_3 only for $\varphi\in(0,\pi/2)$, because for $\varphi\in(-\pi/2,0)$ the value of I_3 can be obtained substituting b instead of a and R instead of r. Let $\varphi\in(0,\pi/2)$. When $\varphi\in\left(0,\arcsin\frac{a}{r}\right)$ for all values of p from the interval $(0,\sqrt{r^2-a^2}\cos\varphi)$ the intersection points lie on different curves.

We have that $P_1 \in \Gamma_1,\ P_2 \in \Gamma_2$, iff $g = (\varphi, p) \in \left(0, \dfrac{\pi}{2}\right) \times \left(0, \sqrt{r^2 - a^2}\cos\varphi\right)$.

To find the value of p, for which the line (φ, p) makes a chord of length z, we have to solve the equation

$$r\sin\left(\varphi + \arccos\frac{p + a\sin\varphi}{r}\right) - R\sin\left(\varphi - \arccos\frac{p - b\sin\varphi}{R}\right) = z\cos\varphi + a + b. \quad (5)$$

3. The Case of a Regular Lens. Consider a particular case, where a lens is regular, that is $r = R,\ a = b$. In this case we can rewrite (5) in the following form:

$$\sqrt{1 - \left(\frac{p + a\sin\varphi}{r}\right)^2} + \sqrt{1 - \left(\frac{p - a\sin\varphi}{r}\right)^2} = \frac{z + 2a\cos\varphi}{r}. \quad (6)$$

Solution of (6) has the form

$$p_z = \left(\frac{z}{2} + a\cos\varphi\right)\sqrt{\frac{4r^2}{4a^2 + z^2 + 4az\cos\varphi} - 1}. \quad (7)$$

Note that for given φ the length of the chord arises by line (φ, p) is maximum when $p = 0$, and minimum when $\sqrt{r^2 - a^2}\cos\varphi$. Thus, if $\varphi \in \left(0, \arcsin\dfrac{a}{r}\right)$, the minimum value of chord length is 0 ($z_{\min}(\varphi) = 0$), and the maximum value is equal to $z_{\max} = 2r\left(\sqrt{1 - \left(\dfrac{a\sin\varphi}{r}\right)^2} - \dfrac{a}{r}\cos\varphi\right)$. Denoting $\dfrac{a}{r} = \lambda$ ($\lambda < 1$), $\dfrac{z}{2r} = u$ $\left(u \in \left(0, \sqrt{1 - \lambda^2}\right)\right)$, we obtain $u_{\max}(\varphi) = \sqrt{1 - \lambda^2\sin^2\varphi} - \lambda\cos\varphi$ and $u_{\min}(\varphi) = 0$. If $\varphi \in \left(\arcsin\lambda, \dfrac{\pi}{2}\right)$, than $u_{\max}(\varphi) = \sqrt{1 - \lambda^2\sin^2\varphi} - \lambda\cos\varphi$, $u_{\min}(\varphi) = \sqrt{1 - \lambda^2}\sin\varphi - \lambda\cos\varphi = \sin(\varphi - \arcsin\lambda)$ and the function $u_{\max}(\varphi) = \sqrt{1 - \lambda^2\sin^2\varphi} - \lambda\cos\varphi$ is an increasing function in the interval $\left(0, \dfrac{\pi}{2}\right)$, $u_{\max}(0) = 1 - \lambda$. Therefore, if $u < 1 - \lambda$, we get

$$I_3 = 4\left[\int_0^{\arcsin\lambda} d\varphi \int_{p_z}^{r\sqrt{1-\lambda^2}\cos\varphi} dp + \int_{\arcsin\lambda}^{\arcsin\lambda + \arcsin u} d\varphi \int_{p_z}^{r\sqrt{1-\lambda^2}\cos\varphi} dp\right] =$$

$$= 4r\left[\int_0^{\arcsin\lambda + \arcsin u} \sqrt{1 - \lambda^2}\cos\varphi d\varphi - \int_0^{\arcsin\lambda + \arcsin u} (u + \lambda\cos\varphi)\sqrt{\frac{1}{\lambda^2 + u^2 + 2\lambda u\cos\varphi} - 1} d\varphi\right] =$$

$$= 4r\left[\sqrt{1 - \lambda^2}\left(u\sqrt{1 - \lambda^2} - \lambda\sqrt{1 - u^2}\right) - \int_{\sqrt{1-\lambda^2}\sqrt{1-u^2} - \lambda u}^{1} \frac{u + \lambda t}{\sqrt{1 - t^2}}\sqrt{\frac{1}{\lambda^2 + u^2 + 2\lambda u t} - 1} dt\right].$$

Similarly, if $1 - \lambda \le u < \sqrt{1 - \lambda^2}$, we have

$$\iint_{A_\xi} d\phi\, dp = 4r\left[\int_0^{\arcsin\lambda+\arcsin u}\sqrt{1-\lambda^2}\cos\varphi\, d\varphi + \int_{\arccos\frac{1-\lambda^2-u^2}{2\lambda u}}^{\arcsin\lambda+\arcsin u}\left(u+\lambda\cos\varphi\right)\sqrt{\frac{1}{\lambda^2+u^2+2\lambda u\cos\varphi}-1}\, d\varphi\right] =$$

$$= 4r\left[\sqrt{1-\lambda^2}\left(u\sqrt{1-\lambda^2}-\lambda\sqrt{1-u^2}\right) - \int_{\sqrt{1-\lambda^2}\sqrt{1-u^2}-\lambda u}^{\frac{1-\lambda^2-u^2}{2\lambda u}}\frac{u+\lambda t}{\sqrt{1-t^2}}\sqrt{\frac{1}{\lambda^2+u^2+2\lambda u t}-1}\, dt\right].$$

Denote $G_{\lambda,u}(x) = \int_{\sqrt{1-\lambda^2}\sqrt{1-u^2}-\lambda u}^{x}\frac{u+\lambda t}{\sqrt{1-t^2}}\sqrt{\frac{1}{\lambda^2+u^2+2\lambda u t}-1}\, dt$.

Finally, for the chord length distribution function for regular lens we obtain

$$F(z)=\begin{cases} 0, \text{ if } u\le 0, \\ \dfrac{1}{\arccos\lambda}\left[\arccos\lambda-\lambda^2 u+\sqrt{1-u^2}\left(\lambda\sqrt{1-\lambda^2}+\arcsin\lambda-\arccos u\right)-G_{\lambda,u}(1)\right], \text{ if } 0\le u<1-\lambda, \\ \dfrac{1}{\arccos\lambda}\left[\arccos\lambda-\lambda^2 u+\sqrt{1-u^2}\left(\lambda\sqrt{1-\lambda^2}+\arcsin\lambda-\arccos u\right)-G_{\lambda,u}\left(\dfrac{1-\lambda^2-u^2}{2\lambda u}\right)\right], \text{ if } 1-\lambda\le u<\sqrt{1-\lambda^2}, \\ 1, \text{ if } u>\sqrt{1-\lambda^2}. \end{cases} \tag{8}$$

It is not difficult to verify that F is a continuous function. When $\lambda\to 0$, the lens D is transformed to a disc of radius r, and from (8) we have

$$F(z)=\begin{cases} 0, & \text{if } u\le 0, \\ 1-\sqrt{1-u^2}, & \text{if } 0\le u\le 1, \\ 1, & \text{if } u>1, \end{cases}$$

which is chord length distribution function for the disc of radius r.

u \ Λ	0.00	0.1	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.00	0.000	0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.05	0.001	0.002	0.003	0.004	0.007	0.012	0.021	0.036	0.070	0.190
0.10	0.005	0.006	0.009	0.013	0.019	0.030	0.049	0.084	0.166	0.569
0.15	0.011	0.014	0.018	0.025	0.036	0.055	0.087	0.148	0.301	0.845
0.20	0.020	0.024	0.031	0.041	0.058	0.086	0.134	0.231	0.550	0.918
0.25	0.032	0.037	0.047	0.062	0.086	0.125	0.195	0.345	0.765	0.955
0.30	0.046	0.054	0.067	0.088	0.120	0.174	0.272	0.545	0.853	0.978
0.35	0.063	0.074	0.091	0.118	0.161	0.233	0.376	0.728	0.906	0.991
0.40	0.083	0.098	0.120	0.155	0.211	0.308	0.551	0.818	0.943	0.998
0.45	0.107	0.125	0.154	0.198	0.270	0.406	0.717	0.879	0.968	
0.50	0.134	0.157	0.193	0.249	0.344	0.567	0.807	0.922	0.986	
0.55	0.165	0.194	0.238	0.310	0.440	0.723	0.870	0.955	0.996	
0.60	0.200	0.236	0.292	0.385	0.594	0.813	0.917	0.978	1.000	
0.65	0.240	0.284	0.355	0.480	0.746	0.878	0.953	0.993		
0.70	0.286	0.341	0.432	0.634	0.835	0.927	0.979	1.000		
0.75	0.339	0.408	0.530	0.786	0.901	0.963	0.994			
0.80	0.400	0.489	0.690	0.875	0.949	0.987	1.000			
0.85	0.473	0.595	0.847	0.939	0.982	0.999				
0.90	0.564	0.773	0.935	0.981	0.999					
0.95	0.688	0.943	0.988	1.000						
1.00	1.000									

In Table values of chord length distribution function are given when λ varies from 0 to 0.9 by step 0.01 and u varies from 0 to $\sqrt{1-\lambda^2}$ by step 0.05.

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