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COMMUNICATIONS

Mathematics

SPLITTING AUTOMORPHISMS OF FREE BURNSIDE GROUPS

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We have proved, that if $n \geq 1003$ is an arbitrary odd number and φ is a splitting automorphism of period n of group $B(m,n)$ that has a prime order, then φ is inner automorphism.

***Keywords*:** splitting automorphism, free Burside group, holomorph of the group.

Definition. An automorphism φ of G is called *splitting automorphism of period* n, if $\varphi^n = 1$ and $g g^{\varphi} g^{\varphi^2} \cdots g^{\varphi^{n-1}} = 1$ for each $g \in G$.

If φ is a splitting automorphism of period n of a group G, then for every $g \in G$ the relation $(\varphi g)^n = 1$ holds in the holomorph $Hol(G)$ of the group G. In particular, the identity automorphism of G is a splitting automorphism of period n, iff the identity $x^n = 1$ is satisfied in G.

A well-known Theorem of O. Kegel states that any finite group that has a nontrivial splitting automorphism of prime period is nilpotent (see [1]). This result generalizes J. Tompson's Theorem [2] on the nilpotency of finite group with an automorphism of prime order without fixed points. E. Khukhro proved that any solvable group having a nontrivial splitting automorphism of prime period is also a nilpotent group. (see [3]).

If Γ is any group satisfying the identical relation $x^n = 1$, then the identities

$$b(a^{-1}ba)(a^{-2}ba^2)\cdots(a^{-n+1}ba^{n-1})^n a^n, \quad a^{-n}ba^n = b$$

also hold in the group Γ. Consequently, each inner automorphism $i_a \in \mathrm{Inn}(\Gamma)$, $i_a(b) = a^{-1}ba$ is a splitting automorphism of period n of the group Γ.

We are interested in the inverse question for splitting automorphisms of the free Burnside group $B(m,n)$. By definition, a free Burnside group $B(m,n)$ of period n and rank m has the following presentation

$$B(m,n) = \langle a_1, a_2, ..., a_m \mid X^n = 1 \rangle,$$

where X runs over all words in $\{a_1^{\pm 1}, a_2^{\pm 1}, \ldots, a_m^{\pm 1}\}$.

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The group $B(m,n)$ is the quotient of the free group F_m of rank m by the normal subgroup F_m^n, generated by all nth powers of elements of F_m. Each periodic group of period n with m generators is a quotient group of $B(m,n)$. According to the fundamental Theorem of S.I. Adian that solves the Burnside problem (see [4]), for any odd $n \geq 665$ and $m > 1$ the group $B(m,n)$ is infinite. A detailed review of studies on free Burnside groups is given by S.I. Adian in [5].

In 1991 S.V. Ivanov posed the following question: let $m > 1$ and n is large enough odd number. Is it true that any splitting automorphism φ of $B(m,n)$ is inner (see Kourovka notebook [6], question 11.36, b)?

We have proved the following theorem.

Theorem. Let $n \geq 1003$ be an arbitrary odd number and φ be a splitting automorphism of period n of $B(m,n)$. If the order of the automorphism φ is a prime number, then φ is inner.

This Theorem immediately implies.

Corollary. For any prime $n > 997$ and $m > 1$ each splitting automorphism of the group $B(m,n)$ is an inner automorphism.

Outline of the Proof of the Main Result. In the paper [7] it was proved that for any $m > 1$ and odd $n \geq 1003$ there exists a maximal normal subgroup N of the free Burnside group $B(m,n)$, such that the quotient $B(m,n)/N$ is an infinite group, every proper subgroup of which is contained in some cyclic subgroup of order n. Denote by $\mathcal{M}_n$ the set of all such maximal normal subgroups N of the free Burnside group $B(m,n)$. The groups $B(m,n)/N$, constructed in [7], are called Tarski monsters. In [8] it was shown that for every odd $n \geq 1003$ there is a continuum of non-isomorphic Tarski monsters of period n.

The following statement, proved by the author in [9], plays a key role in the proof of the main result.

Proposition 1. (see [9], Corollary 2) Let $n \geq 1003$ be an arbitrary odd number and φ be an automorphism of the group $B(m,n)$, such that $\varphi(N) = N$ for any $N \in \mathcal{M}_n$. Then φ is inner automorphism.

The following interesting results obtained by the author are also used in the proof.

Proposition 2. Let $\varphi : G \to G$ be an arbitrary automorphism and N be a normal subgroup of G, such that the quotient G/N is a non-abelian simple group. If the subgroups $N, \varphi(N), \cdots, \varphi^{k-1}(N)$ are pairwise distinct and $\varphi^k(N) = N$, then the quotient group $G / \bigcap_{i=1}^{k} \varphi^i(N)$ is decomposed into a direct product of subgroups $N_j / \bigcap_{i=1}^{k} \varphi^i(N)$, $j = 1,2,...,k$, where $N_j = \bigcap_{\substack{i=1 \\ i \neq j}}^{k} \varphi^i(N)$. Moreover the quotient $N_j / \bigcap_{i=1}^{k} \varphi^i(N)$ is isomorphic to G/N.

Proposition 3. If $n \geq 1003$ is an odd number and φ is an arbitrary nontrivial splitting automorphism of period n of $B(m,n)$, then for every normal subgroup $N \in \mathcal{M}_n$ the stabilizer relative to action of the cyclic group $\langle \varphi \rangle$ is nontrivial.

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