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Mathematics

GENERAL CLASSES OF TAYLOR–MACLAURIN TYPE FORMULAS IN COMPLEX DOMAIN

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In the present paper the results of [1] are generalized, more general systems of operators generated by Riemann–Liouville integral and derivative and general systems of functions generated by Mittag–Leffler type functions are introduced as well.

***Keywords*:** Riemann–Liouville operators, Taylor–Maclaurin's type formula.

§ 1. Preliminaries and Lemmas. Let $\alpha\in[0,1)$, $1/\rho=1-\alpha$ $(\rho\geq 1)$, $f(z)$ is a complex function satisfying $f(re^{i\varphi})\in L(0,l(\varphi))$ for fixed $\varphi(-\pi<\varphi\leq\pi)$, $0<r<+\infty$, where $(0,l(\varphi))=\{z;\ \arg z=\varphi,\ 0<|z|<l<+\infty\}$.

Then the function $D^{-\alpha}f(z)=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^z(z-\xi)^{\alpha-1}f(\xi)d\xi$, where the integration is taken along the line segment connecting points 0 and z, $\arg(z-\xi)^{\alpha-1}=(\alpha-1)\arg z$, is called the Riemann–Liouville integral of order α of $f(z)$, and the function $D^{1/\rho}f(z)\equiv\dfrac{d}{dz}D^{-\alpha}f(z)$ is called the Riemann–Liouville derivative of order $1/\rho$ of $f(z)$.

The operators $D^{0/\rho}f(z)=f(z)$, $D^{1/\rho}f(z)$, $D^{n/\rho}f(z)=D^{1/\rho}D^{(n-1)/\rho}f(z)$, $n\geq 2$, are called the Riemann–Liouville operators of successive differentiation of order n/ρ, $n=0,1,...,$ of function $f(z)$.

It is known, that if $\alpha\in(0,1)$, then $D^{\alpha}f(z)\equiv\dfrac{d}{dz}D^{-(1-\alpha)}f(z)$.

If $f(x)\in L(0,l)$, then for any $\alpha\in(0,+\infty)$, a.e. $D^{\alpha}D^{-\alpha}f(x)=f(x)$. If $\alpha\in(0,1]$, $D^{\alpha}f(x)\in L(0,l)$ and $[D^{-(1-\alpha)}f(x)]_{x=0}=0$, then a.e. $D^{-\alpha}D^{\alpha}f(x)=f(x)$ (see [2], chap. IX, (1.11), (1.12')). These formulas are true also for complex variable z.

Let $\mu>0,\ \alpha>0$. Then the following formula is true:

$$\frac{1}{\Gamma(\alpha)}\int_0^z(z-\xi)^{\alpha-1}E_{\rho}(\lambda\xi^{1/\rho};\mu)\xi^{\mu-1}d\xi=z^{\mu+\alpha-1}E_{\rho}(\lambda z^{1/\rho};\mu+\alpha)\qquad(1.1)$$

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(see [2], chap. III, (1.16)), where λ is an arbitrary parameter, the integration is taken along the line segment connecting points 0 and z, and $E_\rho(z;\mu)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+k/\rho)}$ $(\rho>0)$ is the ρ order entire function of Mittag–Leffler type for any value of parameter μ. We put

$$e_{\rho,\mu}(z;\lambda)\equiv E_\rho(-\lambda z^{1/\rho};\mu)z^{\mu-1},\ \ e_{\rho,1/\rho}(z;\lambda)=E_\rho(-\lambda z^{1/\rho};1/\rho)z^{1/\rho-1}\equiv e_\rho(z;\lambda).$$

Lemma 1.1. Let $\rho\geq 1,\ 1/\rho\leq\mu<1+1/\rho$, λ is an arbitrary parameter. Then the following formula holds:

$$D^{(\mu-1/\rho)}\{e_{\rho,\mu}(z;\lambda)\}=e_\rho(z;\lambda). \tag{1.2}$$

Proof. Since $0\leq\mu-1/\rho<1$, then according to the definition of operator $D^{(\mu-1/\rho)}$, using formula (1.1) we have

$$D^{(\mu-1/\rho)}\{e_{\rho,\mu}(z;\lambda)\}=\frac{d}{dz}D^{-(1+1/\rho-\mu)}\{e_{\rho,\mu}(z;\lambda)\}=$$

$$=\frac{d}{dz}\left\{\frac{1}{\Gamma(1+1/\rho-\mu)}\int_0^z(z-\xi)^{1/\rho-\mu}E_\rho(-\lambda\xi^{1/\rho};\mu)\xi^{\mu-1}d\xi\right\}=$$

$$=\frac{d}{dz}\{E_\rho(-\lambda z^{1/\rho};1+1/\rho)z^{1/\rho}\}=E_\rho(-\lambda z^{1/\rho};1/\rho)z^{1/\rho-1}=e_\rho(z;\lambda),$$

$$z=re^{i\varphi},\ \xi=\tau e^{i\varphi},\ 0<\tau<r<l<+\infty.$$

Lemma 1.2. Let $\rho\geq 1,\ 1/\rho\leq\mu<1+1/\rho$, λ is an arbitrary parameter. Then the following formula holds:

$$D^{-(\mu-1/\rho)}D^{(\mu-1/\rho)}\{e_{\rho,\mu}(z;\lambda)\}=e_{\rho,\mu}(z;\lambda). \tag{1.3}$$

Lemma 1.3. Let $\rho\geq 1,\ 1/\rho\leq\mu<1+1/\rho$, λ is an arbitrary parameter, $f(z)\in L(0;l(\varphi))$, $-\pi<\varphi\leq\pi$, is continuous on $(0;l(\varphi))$. Then the following formula holds:

$$D^{(\mu-1/\rho)}\left\{\int_0^z e_{\rho,\mu}(z-\xi;\lambda)f(\xi)d\xi\right\}=\int_0^z e_\rho(z-\xi;\lambda)f(\xi)d\xi,$$
$$z=re^{i\varphi},\ \ \xi=\tau e^{i\varphi},\ 0<\tau<r<l<+\infty. \tag{1.4}$$

Lemma 1.4. Let $\rho\geq 1,\ 1/\rho\leq\mu<1+1/\rho$, λ is an arbitrary parameter, $f(z)\in L(0;l(\varphi))$ is continuous on $(0;l(\varphi))$. Then the following formula holds:

$$D^{-(\mu-1/\rho)}\left\{\int_0^z e_\rho(z-\xi;\lambda)f(\xi)d\xi\right\}=\int_0^z e_{\rho,\mu}(z-\xi;\lambda)f(\xi)d\xi,$$
$$z=re^{i\varphi},\ \ \xi=\tau e^{i\varphi},\ 0<\tau<r<l<+\infty. \tag{1.5}$$

We note that these Lemmas for $z=x\in(0,+\infty)$ are proven in [4]. We remark also that these Lemmas are true for $\mu\geq 1+1/\rho$, but then $D^{(\mu-1/\rho)}f(z)=$ $=\frac{d^P}{dz^P}\{D^{-(P+1/\rho-\mu)}f(z)\}$, where P is an integer satisfying $P-1<\mu-1/\rho\leq P$.

Lemma 1.5. Let $\alpha\in[0,1)$, $1/\rho=1-\alpha$ $(\rho\ge 1)$, $1/\rho\le\mu<1+1/\rho$, λ is an arbitrary parameter. Then in the class of functions satisfying $D^{(\mu-1/\rho)}y(re^{i\varphi})\in L(0,l(\varphi))$, $D^{-\alpha}\{D^{(\mu-1/\rho)}y(re^{i\varphi})\}\in L(0,l(\varphi)), -\pi<\varphi\le\pi$, the following of Cauchy type problem

$$(D^{1/\rho}+\lambda)D^{(\mu-1/\rho)}y(z)=0, \qquad D^{-\alpha}\{D^{(\mu-1/\rho)}y(z)\}\big|_{z=0}=0,$$
$$D^{-(1+1/\rho-\mu)}\,y(z)\big|_{z=0}=0 \tag{1.6}$$

has a unique solution $y(z)\equiv 0$.

Proof. We put $D^{(\mu-1/\rho)}y(z)\equiv\tilde y(z)$. Then the problem (1.6), according to Theorem 2.1 [5], has a unique solution $\tilde y(z)\equiv 0$. Now we not that

$$D^{(\mu-1/\rho)}y(z)=0\,. \tag{1.7}$$

If we apply the operator $\tilde D^{(\mu-1/\rho)}$ to (1.7), then, using (1.6), we will have $y(z)\equiv 0$.

Lemma 1.6. Let $\alpha\in[0,1)$, $1/\rho=1-\alpha$ $(\rho\ge 1)$, $1/\rho\le\mu<1+1/\rho$, λ is an arbitrary parameter, the function $f(re^{i\varphi})\in L(0,l(\varphi))$ is continuous on $(0,l(\varphi))$, $-\pi<\varphi\le\pi$. Then in the class of functions satisfying $D^{(\mu-1/\rho)}y(re^{i\varphi})\in L(0,l(\varphi))$, $D^{1/\rho}\{D^{(\mu-1/\rho)}y(re^{i\varphi})\}\in L(0,l(\varphi))$ the following Cauchy type problem

$$(D^{1/\rho}+\lambda)D^{(\mu-1/\rho)}y(z)=f(z), \qquad D^{-\alpha}\{D^{(\mu-1/\rho)}y(z)\}\big|_{z=0}=0,$$
$$D^{-(1+1/\rho-\mu)}\,y(z)\big|_{z=0}=0, \tag{1.8}$$

has a unique solution

$$y(z)=\int_0^z e_{\rho,\mu}(z-\xi;\lambda)f(\xi)d\xi,\ z=re^{i\varphi},\ \xi=\tau e^{i\varphi},\ 0<\tau<r<l<+\infty\,. \tag{1.9}$$

Proof. We put $D^{(\mu-1/\rho)}y(z)=\tilde y(z)$. Then we get the following Cauchy type problem

$$(D^{1/\rho}+\lambda)\tilde y(z)=f(z),\quad D^{-\alpha}\,\tilde y(z)\big|_{z=0}=0\,. \tag{1.10}$$

According to Theorem 2.3 from [5], the problem (1.10) has a unique solution

$$\tilde y(z)=\int_0^z e_\rho(z-\xi;\lambda)f(\xi)d\xi\,. \tag{1.11}$$

We apply operator $\tilde D^{(\mu-1/\rho)}$ to (1.11). Then, by the Lemmas 1.2–1.4, we get $D^{-(\mu-1/\rho)}\tilde y(z)=D^{-(\mu-1/\rho)}D^{(\mu-1/\rho)}y(z)=D^{-(\mu-1/\rho)}\left\{\int_0^z e_\rho(z-\xi;\lambda)f(\xi)d\xi\right\}$, i.e.

$$y(z)=\int_0^z e_{\rho,\mu}(z-\xi;\lambda)f(\xi)d\xi.$$

§ 2. Main Results. Let sequences $\{\lambda_j\}_0^\infty$, $\{\rho_j\}_0^\infty$, $\{\mu_j\}_0^\infty$ satisfy the conditions

$$\rho_j\ge 1,\quad 1/\rho_j\le\mu_j<1+1/\rho_j,\quad 0<\lambda_j<\lambda_{j+1},\quad j=0,1,... \tag{2.1}$$

Consider the sequence of operators on an admissible class of functions $f(z)$, $\{\Delta^{(n)}f(z)\}_0^\infty$, $\{\tilde\Delta^{(n)}f(z)\}_0^\infty$ defined by

$$\Delta^{(0)} f(z) \equiv f(z), \ \Delta^{(n)} f(z) = \left\{ \prod_{j=0}^{n-1} \tilde{D}_j \right\} f(z), \ n \geq 1, \tag{2.2}$$

$$\tilde{\Delta}^{(n)} f(z) \equiv D^{(-\alpha_n)} D^{(\mu_n - 1/\rho_n)} \Delta^{(n)} f(z), \ n \geq 0, \tag{2.3}$$

where $(j \geq 0)$

$$\tilde{D}_j f(z) \equiv (D^{1/\rho_j} + \lambda_j) D^{(\mu_j - 1/\rho_j)} f(z), \qquad D^{-\alpha_j} f(z) \equiv \frac{1}{\Gamma(\alpha_j)} \int_0^z (z-\xi)^{\alpha_j - 1} f(\xi) d\xi,$$

$$D^{1/\rho_j} f(z) \equiv \frac{d}{dz} D^{-\alpha_j} f(z), \ \alpha_j = 1 - (1/\rho_j), \ D^{(\mu_j - 1/\rho_j)} f(z) \equiv \frac{d}{dz} D^{-(1+1/\rho_j - \mu_j)} f(z).$$

Consider the sequence of functions $\{\Omega_n(z; \{\lambda_j, \rho_j, \mu_j\}_0^n)\}_0^\infty$ given by

$$\Omega_0(z; \{\lambda_0, \rho_0, \mu_0\}) \equiv e_{\rho_0, \mu_0}(z; \lambda_0),$$

$$\Omega_1(z; \{\lambda_j, \rho_j, \mu_j\}_0^1) \equiv \int_0^z e_{\rho_0, \mu_0}(z - t_1; \lambda_0) e_{\rho_1, \mu_1}(t_1; \lambda_1) dt_1,$$

$$\begin{aligned} \Omega_n(z; \{\lambda_j, \rho_j, \mu_j\}_0^n) \equiv \int_0^z e_{\rho_0, \mu_0}(z - t_1; \lambda_0) dt_1 \int_0^{t_1} e_{\rho_1, \mu_1}(t_1 - t_2; \lambda_1) dt_2 \times \\ \times \int_0^{t_{n-1}} e_{\rho_{n-1}, \mu_{n-1}}(t_{n-1} - t_n; \lambda_{n-1}) e_{\rho_n, \mu_n}(t_n; \lambda_n) dt_n, \ n \geq 2, \end{aligned} \tag{2.4}$$

where $z = re^{i\varphi}$, $t_k = \tau_k e^{i\varphi}$ $(k = 0, 1, ..., n)$, $0 < \tau_n < \tau_{n-1} < ... < \tau_1 < r < l < +\infty$,

$$e_{\rho_j, \mu_j}(z; \lambda_j) \equiv E_{\rho_j}(-\lambda_j z^{1/\rho_j}; \mu_j) z^{\mu_j - 1} = \sum_{k=0}^{\infty} \frac{(-\lambda_j)^k z^{k/\rho_j}}{\Gamma(\mu_j + k/\rho_j)} z^{\mu_j - 1}, \ j = 0, 1, ...$$

Note that for $\rho_j = \rho \geq 1$, $\mu_j = 1/\rho$, $j = 0, 1, ...$, operators (2.2), (2.3) and the system of functions (2.4) were introduced in [1], and for $z = x \in (0, +\infty)$ – in [3]. For $\rho_j \geq 1$, $\mu_j \geq 1/\rho_j$, $j = 0, 1, ...$, $z = x \in (0, +\infty)$ these systems were introduced in [4].

Lemma 2.1.

1^0. For any $n \geq 0$ the following relations hold:

$$\Delta^{(k)} \{\Omega_n(z; \{\lambda_j; \rho_j; \mu_j\}_0^n)\} = \tilde{\Delta}^{(k)} \{\Omega_n(z; \{\lambda_j; \rho_j; \mu_j\}_0^n)\} \equiv 0, \ k \geq n+1, \tag{2.5}$$

$$\tilde{\Delta}^{(n)} \{\Omega_n(z; \{\lambda_j; \rho_j; \mu_j\}_0^n)\} = E_{\rho_n}(-\lambda_n z^{1/\rho_n}; 1). \tag{2.6}$$

2^0. $$\tilde{\Delta}^{(k)} \{\Omega_n(z; \{\lambda_j; \rho_j; \mu_j\}_0^n)\}\Big|_{z=0} = 0, \ 0 \leq k \leq n-1. \tag{2.7}$$

3^0. $$D^{-(1+1/\rho_k - \mu_k)} \Delta^{(k)} \{\Omega_n(z; \{\lambda_j; \rho_j; \mu_j\}_0^n)\}\Big|_{z=0} = 0, \ 0 \leq k \leq n. \tag{2.8}$$

The parts 1^0, 2^0 of Lemma 2.1 can be proved in the same way as the Lemma 3.1 for $z = x \in (0, +\infty)$ (see [4]). So, we need to prove only (2.8). For $0 < k < n$ denote

$$\Omega_{k,n}(z; \{\lambda_j, \rho_j, \mu_j\}_k^n) = \int_0^z e_{\rho_k, \mu_k}(z - t_{k+1}; \lambda_k) dt_{k+1} \int_0^{t_{k+1}} e_{\rho_{k+1}, \mu_{k+1}}(t_{k+1} - t_{k+2}; \lambda_{k+1}) dt_{k+2} ... \times$$

$$\times ... \times \int_0^{t_{n-1}} e_{\rho_{n-1}, \mu_{n-1}}(t_{n-1} - t_n; \lambda_{n-1}) e_{\rho_n, \mu_n}(t_n; \lambda_n) dt_n, \qquad z = re^{i\varphi}, \ t_{k+1} = \tau_{k+1} e^{i\varphi}, ...,$$

$$t_n = \tau_n e^{i\varphi}, \ \tau_n < \tau_{n-1} < ... < t_{k+1} < r < +\infty.$$

$\Omega_{0,n}(z;\{\lambda_j,\rho_j,\mu_j\}_0^n)\equiv\Omega_n(z;\{\lambda_j,\rho_j,\mu_j\}_0^n)$, $\Omega_{n,n}(z;\{\lambda_n,\rho_n,\mu_n\})\equiv e_{\rho_n,\mu_n}(z;\lambda_n)$.

Let $0\le k\le n-1$. Using the definition of operator $\Delta^{(k)}$ and Lemma 1.6, we easily obtain

$$\Delta^{(k)}\{\Omega_n(z;\{\lambda_j,\rho_j,\mu_j\}_0^n)\}=\Omega_{k,n}(z;\{\lambda_j,\rho_j,\mu_j\}_k^n). \tag{2.9}$$

Now applying the operator $D^{-(1+1/\rho_k-\mu_k)}$ to (2.15), we get

$$D^{-(1+1/\rho_k-\mu_k)}\Delta^{(k)}\{\Omega_n(z;\{\lambda_j,\rho_j,\mu_j\}_0^n)\}=D^{-(1+1/\rho_k-\mu_k)}\{\Omega_{k,n}(z;\{\lambda_j,\rho_j,\mu_j\}_k^n)\}=$$
$$=\int_0^z e_{\rho_k,\,1+1/\rho_k}(z-t_{k+1};\lambda_k)\Omega_{k+1,n}(t_{k+1};\{\lambda_j,\rho_j,\mu_j\}_{k+1}^n)dt_{k+1}. \qquad 2.10)$$

For $k=n$ we will have

$$D^{-(1+1/\rho_n-\mu_n)}\Delta^{(n)}\{\Omega_n(z;\{\lambda_j,\rho_j,\mu_j\}_0^n)\}=E_{\rho_n}(-\lambda_n z^{1/\rho_n};1+1/\rho_n)z^{1/\rho_n}. \tag{2.11}$$

From (2.10), (2.11) the formula (2.8) follows.

Lemma 2.2. Let the function $f(z)\in L(0,l(\varphi))$ be continuous on $(0,l(\varphi))$, $0<l<+\infty$. Then for any $n\ge1$ the following formula holds

$$\int_0^z e_{\rho_0,\mu_0}(z-t_1;\lambda_0)dt_1\int_0^{t_1}e_{\rho_1,\mu_1}(t_1-t_2;\lambda_1)dt_2\ldots\int_0^{t_{n-1}}e_{\rho_{n-1},\mu_{n-1}}(t_{n-1}-t_n;\lambda_{n-1})dt_n\int_0^{t_n}e_{\rho_n,\mu_n}(t_n-t_{n+1};\lambda_n)f(t_{n+1})dt_{n+1}=$$
$$=\int_0^z\Omega_n(z-t_1;\{\lambda_j,\rho_j,\mu_j\}_0^n)f(t_1)dt_1,\ z=re^{i\varphi},\ t_k=\tau_k e^{i\varphi},\ k=0,1,\ldots,n+1,\ \tau_{n+1}<\tau_n<\ldots<\tau_1<r<+\infty.$$

Lemma 2.2 can be proved in the same way as Lemma 4.1 from [4] for $z=x\in(0,+\infty)$.

Lemma 2.3. For any $n\ge0$ the coefficients $\{a_k\}_0^n$ of the sum

$$P_n(z)=\sum_{k=0}^n a_k\Omega_k(z;\{\lambda_j,\rho_j,\mu_j\}_0^k) \tag{2.12}$$

can be recovered by the formula $a_k=\tilde\Delta^{(k)}P_n(0),\quad k=0,1,\ldots,n$.

Proof. Let $0\le j\le n-1$. Applying the operator $\tilde\Delta^{(j)}$ to function $P_n(z)$ and using (2.5)–(2.8), we obtain

$$\tilde\Delta^{(j)}P_n(z)=a_jE_{\rho_j}(-\lambda_j z^{1/\rho_j};1)+\sum_{k=j+1}^n a_k\Omega_k(z;\{\lambda_j,\rho_j,\mu_j\}_0^k). \tag{2.13}$$

But since $E_{\rho_j}(0;1)=1$, then from (2.13) and (2.7) we obtain $a_j=\tilde\Delta^{(j)}P_n(0)$, $j=0,1,\ldots,n-1$. Now we apply the operator $\tilde\Delta^{(n)}$ to (2.12): $\tilde\Delta^{(n)}P_n(z)=a_n\tilde\Delta^{(n)}\{\Omega_n(z;\{\lambda_j,\rho_j,\mu_j\}_0^n)\}=a_nE_{\rho_n}(-\lambda_n z^{1/\rho_n};1)$, hence $a_n=\tilde\Delta^{(n)}P_n(0)$.

Lemma 2.3 is proved.

Lemma 2.4. Let $P_n(z)=\sum_{m=0}^n a_m\Omega_m(z;\{\lambda_j,\rho_j,\mu_j\}_0^m)$. Then for any $n\ge0$

$$D^{-(1+1/\rho_k-\mu_k)}\Delta^{(k)}\{P_n(z)\}\big|_{z=0}=0,\quad k=0,1,\ldots,n. \tag{2.14}$$

We note that from (2.5)–(2.8) it is easy to obtain (2.14). Now denote by $C_{n+1}\{[0,l(\varphi)),<\lambda_j>,<\rho_j>,<\mu_j>\}$ the set of functions $f(z)$ satisfying the following conditions:

1) the functions $\tilde{\Delta}^{(k)} f(z),\ k=0,1,...,n$ are continuous on $[0,l(\varphi))$, $-\pi<\varphi\le\pi$;

2) the functions $\Delta^{(k)} f(z)\ (k=0,1,...,n+1)$ are continuous on $[0,l(\varphi))$ and $\Delta^{(k)} f(z)\in L(0,l(\varphi))$;

3) $[D^{-(1+1/\rho_k-\mu_k)}\Delta^{(k)} f(z)]\Big|_{z=0}=0,\quad k=0,1,...n$. (2.15)

Note that any function $P_n(z)=\sum_{k=0}^{n} a_k\Omega_k(z;\{\lambda_j,\rho_j,\mu_j\}_0^k)$ belongs to $C_{n+1}\{[0,l(\varphi)),<\lambda_j>,<\rho_j>,<\mu_j>\}$.

Theorem. If $f(z)\in C_{n+1}\{[0,l(\varphi)),<\lambda_j>,<\rho_j>,<\mu_j>\}$, then

$$f(z)=\sum_{k=0}^{n}\tilde{\Delta}^{k} f(0)\Omega_k(z;\{\lambda_j,\rho_j,\mu_j\}_0^k)+R_n(z), \tag{2.16}$$

where

$$R_n(z)=\int_0^z \Omega_n(z-\xi;\{\lambda_j,\rho_j,\mu_j\}_0^n)\Delta^{(n+1)} f(\xi)d\xi,$$
$$z\in(0,l(\varphi)),\ -\pi<\varphi\le\pi,\quad z=re^{i\varphi},\ \xi=\tau e^{i\varphi},\quad 0<\tau<r<+\infty. \tag{2.17}$$

Proof. Let $P_n(z)=\sum_{k=0}^{n}\tilde{\Delta}^{(k)} f(0)\Omega_k(z;\{\lambda_j,\rho_j,\mu_j\}_0^k)$, $R_n(z)=f(z)-P_n(z)$.

We note that according to Lemmas 2.1, 2.3 and 2.4, from (2.15) we deduce that function $R_n(z)$ satisfies the following conditions:

$$\Delta^{(n+1)}R_n(z)=\Delta^{(n+1)} f(z)\equiv\psi(z),\quad z\in(0,l(\varphi)), \tag{2.18}$$

$$[\tilde{\Delta}^{(k)}R_n(z)]\Big|_{z=0}=0,\quad k=0,1,...,n, \tag{2.19}$$

$$[D^{-(1+1/\rho_k-\mu_k)}R_n(z)]\Big|_{z=0}=0,\quad k=0,1,...,n. \tag{2.20}$$

Using the definition of operator $\Delta^{(n+1)}$ we can write (2.18) in the form

$$(D^{1/\rho_n}+\lambda_n)D^{(\mu_n-1/\rho_n)}\Delta^{(n)}R_n(z)=\psi(z). \tag{2.18'}$$

Note that the function $\Delta^{(n)}R_n(z)$ satisfies the conditions of Lemma 1.6, so, the function $\Delta^{(n)}R_n(z)$ is uniquely determined from (2.18') by means of the integral formula

$$\Delta^{(n)}R_n(z)=\int_0^z e_{\rho_n,\mu_n}(z-t_{n+1};\lambda_n)\psi(t_{n+1})dt_{n+1},\ z=re^{i\varphi},\ t_{n+1}=\tau_{n+1}e^{i\varphi},\ 0<\tau_{n+1}<r<l<+\infty.$$

Further we consider the equation

$$(D^{(1/\rho_{n-1})}+\lambda_{n-1})D^{(\mu_{n-1}-1/\rho_{n-1})}\Delta^{(n-1)}R_n(z)=\int_0^z e_{\rho_n,\mu_n}(z-t_{n+1};\lambda_n)\psi(t_{n+1})dt_{n+1}.$$

Using Lemma 1.6 and (2.19), (2.20), we get

$$\Delta^{(n-1)}R_n(z)=\int_0^z e_{\rho_{n-1},\mu_{n-1}}(z-t_n;\lambda_{n-1})dt_n\int_0^{t_n} e_{\rho_n,\mu_n}(t_n-t_{n+1};\lambda_n)\psi(t_{n+1})dt_{n+1}.$$

Repeating our argument successively exhausting all the initial conditions (2.19), (2.20), we arrive at the identity

$$R_n(z)=\int_0^z e_{\rho_0,\mu_0}(z-t_1;\lambda_0)dt_1\int_0^{t_1} e_{\rho_1,\mu_1}(t_1-t_2;\lambda_1)dt_2\times$$

$$\times\int_0^{t_n} e_{\rho_n,\mu_n}(t_n-t_{n+1};\lambda_n)\psi(t_{n+1})dt_{n+1}=\int_0^z \Omega_n(z-t_1;\{\lambda_j,\rho_j,\mu_j\}_0^n)\Delta^{(n+1)}f(t_1)dt_1,$$

$$z=re^{i\varphi},\ t_j=\tau_j e^{i\varphi},\ \ \tau_{n+1}<\tau_n<\tau_1<r<l<+\infty.$$

Theorem is proved.

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