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ON THE INDEPENDENCE NUMBERS OF THE POWERS OF C_5 GRAPH

S. H. BADALYAN*, A. Zh. MNATSAKANYAN

Chair of Discrete Mathematics and Theoretical Computer Science YSU, Armenia

In this paper independence numbers of the powers of C_5 graph is investigated. Independence number of the 3rd degree of C_5 is calculated and a method is given that can help calculate independence numbers of higher degrees of C_5. Independence number of the 3rd degree of C_5 is also calculated by the given method.

***Keywords*:** independence number, powers of odd cycles, Shannon capacity.

Introduction. Strong product of given two graphs G_1 and G_2 is defined as a graph G that has a vertex set $V(G_1)\times V(G_2)$ and two distinct vertices (v_1,v_2), (u_1,u_2) are connected iff they are adjacent or equal in each coordinate. Since the strong product is associative and commutative we can naturally define G^k. In [1] Shannon introduced the parameter $c(G)=\sup_k \sqrt[k]{\alpha(G^k)}$, the Shannon capacity of graph G, where $\alpha(G^k)$ is the independence number of G^k.

Calculating the Shannon capacity (motivated by Information Theory) is considered very difficult and the problem remains open even for such a simple graph as C_7. The best known upper bounds on the Shannon capacities of graphs are given by the Lovasz theta function [2]. The upper bound suffices to establish the Shannon capacity of C_5 without actually determining independence numbers of its powers $c(C_5)=\sqrt{5}$. In this paper we go from the opposite side trying to calculate independence numbers of powers of C_5 with the hope that it will also have some contribution in finding independence numbers of powers of odd cycles in general (particularly for C_7) and, therefore, in calculating the Shannon capacity for odd cycles.

The Shannon capacities of odd cycles on seven or more vertices remain unknown. Currently, the best known lower bound [3] for $c(C_7)$ is achieved by constructing an independent set of 108 vertices in the 4th power of C_7 (i.e. $c(C_7)\ge\sqrt[4]{108}$).

* E-mail: sevak_badalyan@yahoo.com

x				
		x		
				x
	x			
			x	

Fig. 1.

Preliminary Facts. We will picture C_5^n graph as a table, cells of which represent graph vertices. In Fig. 1 is the C_5^2 graph with one of its maximal independent sets marked.

The 25 cells correspond to the graph vertices and two vertices are adjacent, iff corresponding cells are in adjacent (or the same) rows and columns (note that the table is cyclic, i.e. first and last rows/columns are considered adjacent).

Let's prove the following facts:

1. $\alpha(C_5) = 2$, $\alpha(C_5^2) = 5$.

Proof. The first equality is straightforward, let's prove the second. Since $\alpha(C_5^2) \geq 5$, it's enough to prove that $\alpha(C_5^2) \leq 5$. Assume the opposite: $\alpha(C_5^2) > 5$ and consider any maximal independent set. In that case there is a row that contains at least 2 vertices of the maximal independent set and, therefore, there can be no vertices of the maximal independent set in the adjacent rows. The remaining 2 rows can contain no more than 2 vertices of the independent set.

Thus, the Statement is proved.

2. Any two maximal independent sets in C_5^2 have at most one common vertex.

Proof. From the proof of the previous Statement it follows that any maximal independent set of C_5^2 has exactly one vertex in each row (column). Having in mind this observation, it can be easily seen that any two vertices of maximal independent set uniquely determine next vertices of the independent set.

3. For each vertex of C_5^2 there are exactly two maximal independent sets passing through that vertex.

Proof. For the given vertex there are two vertices in the next row that are not adjacent with the initial vertex. The given vertex with each of those two vertices uniquely identifies a maximal independent set in C_5^2.

4. $2\alpha(C_5^n) \leq \alpha(C_5 \times C_5^n) \leq 2.5\alpha(C_5^n)$.

Proof. The first inequality is clear. The second one follows from the following inequality obtained by Hales [4]: $\alpha(G \times H) \leq \rho(G)\alpha(H)$, where $\rho(G)$ is the Rosenfeld number [5] of graph G (note that $\rho(C_5) = 2.5$).

5. $5\alpha(C_5^n) \leq \alpha(C_5^2 \times C_5^n)$ and, therefore, $\alpha(C_5^{2n}) \geq 5^n$.

Proof. The inequality is a direct consequence of the following: $\alpha(G \times H) \geq \alpha(G)\alpha(H)$.

The following result is obtained in [2]: $\sup\limits_n \left(\sqrt[n]{\alpha(C_5^n)}\right) = \sqrt{5}$. Taking into account the 5th property above, we get $\alpha(C_5^{2n}) = 5^n$. Thus:

6. $\alpha(C_5^{2n}) = 5^n$.

7. $\alpha(C_5^{2n+1}) \geq 2 \cdot 5^n$.

It is proved in [6], that $\alpha(C_5^3) = 10$. Below we give a proof similar to the one in [6], while in the end we use a new method for calculating $\alpha(C_5^3)$. The method can be helpful for finding independence numbers of higher degrees of C_5. Independence numbers for odd cycles of C_5 starting from 5th power are not known. It has been conjectured by different authors that $\alpha(C_5^{2n+1}) = 2\cdot 5^n$ (see, for example, [6]).

Independence Number of C_5^3. Since $C_5^3 = C_5^2 \times C_5$, we can imagine it as a "5-cycle" of C_5^2 graphs, where the subgraph induced by 2 adjacent C_5^2 graphs is $C_5^2 \times K_2$. Thus, mentioned C_5^2 subgraphs can be enumerated, so that "adjacent" C_5^2 subgraphs take consecutive numbers (except first and last subgraphs, which are also adjacent).

Let's determine $\alpha(C_5^3)$. Clearly, each maximal independent set S of C_5^3 will be divided between above mentioned 5 subgraphs. Denote corresponding independent sets in C_5^2 subgraphs: $S_1, S_2, ..., S_5$. It's clear that $\alpha(C_5^3) \geq 10$. We'll prove that $\alpha(C_5^3) = 10$. Suppose, there is an independent set S with cardinal number greater than 10. In that case one of the following observations takes place:

- $\exists i$ (without loss of generality i=1) $|S_1| = 5$. In that case $|S_2| = |S_5| = 0$ and $|S| \leq 10$.
- $\exists i$ (without loss of generality i=1) $|S_1| = 4$. Since $|S| > 10$, then $|S_2| = |S_5| = 1$ and by the second property above $S_2 = S_5$. Therefore, $S_2 \cup S_3 \cup S_4$ is an independent set in C_5^2 graph. Thus, $|S_3| + |S_4| < 5$ and $|S| \leq 10$.
- For 3 consecutive independent components (without loss of generality assume first 3 components) takes place: $|S_1| = 2, |S_2| = 3, |S_3| = 2$. According to the second property above $S_1 = S_3$. In that case $S_1 \cup S_3 \cup S_4 \cup S_5$ is an independent set and, therefore, $|S_4| + |S_5| \leq 3$. Thus, $|S| \leq 10$.
- For 4 consecutive independent components (without loss of generality assume first 4 components) takes place: $|S_1| = 3, |S_2| = 2, |S_3| = 2, |S_4| = 3$. It can be seen that each vertex of S_5 is adjacent to each vertex in S_2 and S_3 (considered in C_5^2 graph). Indeed, if any vertex v of S_5 is not adjacent to any vertex of S_2 (S_3), then substituting the other vertex of S_2 (S_3) with v in maximal independent set $S_1 \cup S_2$ ($S_3 \cup S_4$) would result in another maximal independent set having more than one vertex in common with $S_1 \cup S_2$, which contradicts to the second property mentioned above. Thus, every vertex of S_5 is adjacent to every vertex of S_2 and S_3 and, therefore, there is at most one vertex in S_5, which has the following disposition with the vertices of S_2 and S_3 (the vertex of S_5 is in the center) (Fig. 2). It is not difficult to see that there is no an independent set of cardinality 5 in C_5^2 containing any two of mentioned 5 vertices. This contradicts to the fact that $S_3 \cup S_4$ is an independent set. Therefore, $S_5 = \varnothing$ and $|S| \leq 10$.

	X		X	
		X		
	X		X	

Fig. 2.

Thus, $\alpha(C_5^3) = 10$.

On a Method of Finding Independence Numbers of Powers of C_5 in the General Case. Assume graph G is given. Let $S_1,S_1,...,S_n$ be any partitioning of vertices of G into any independent sets. Consider graph S with vertices $S_1,S_1,...,S_n$ and two vertices S_i, S_j are adjacent, iff there is an edge in G connecting any vertex of S_i to any vertex of S_j. Let's call G an independent extension of graph S.

Theorem 1. For every independent set of C_5^{2n+1} there is a subgraph in $C_5^{2n} \times K_2$ with the same cardinality, which is an independent extension for some subgraph of C_5.

Proof. Assume S is any independent set in C_5^{2n+1}. As mentioned above, C_5^{2n+1} can be represented as a "5-cycle" of C_5^{2n} graphs. Denote intersections of S with those subgraphs $S_1,S_1,...,S_5$ correspondingly. We will consider vertices of S_i in the context of C_5^{2n} graph. Now, let's construct corresponding subgraph of $C_5^{2n} \times K_2$, which is an independent extension for some subgraph of C_5. It's sufficient to indicate corresponding two components of the subgraph in C_5^{2n} graph, since vertex set of $C_5^{2n} \times K_2$ is a combination of vertices of two C_5^{2n} graphs. As such components consider subgraphs in C_5^{2n} induced by $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ and $(S_1 \cap S_3) \cup (S_2 \cap S_4) \cup (S_3 \cap S_5) \cup (S_4 \cap S_1) \cup (S_5 \cap S_2)$. One can check that indicated subgraph satisfies conditions of the Theorem.

The Theorem is proved.

Analogously it can be obtained that the opposite statement is also true, i.e. for each subgraph of $C_5^{2n} \times K_2$, satisfying the conditions of the Theorem, there is an independent set in C_5^{2n+1} with the same cardinality. It means, in order to find maximal independent set (or independence number) of C_5^{2n+1} it suffices to find corresponding maximal subgraph (number of vertices of the subgraph) of $C_5^{2n} \times K_2$.

According to the 7th property above $\alpha(C_5^{2n+1}) \geq 2 \cdot 5^n$. The following statement is true:

Theorem 2. If S is an independent set of C_5^{2n+1} and $|S| > 2 \cdot 5^n$, then any subgraph in $C_5^{2n} \times K_2$ corresponding to S is not an independent extension of any proper subgraph of C_5.

Proof. Assume the opposite and consider any subgraph G in $C_5^{2n} \times K_2$ corresponding to S. Since it is an independent extension for some proper subgraph of C_5, then it doesn't contain odd cycles. Therefore, it is a bipartite graph, each partition of which is an independent set in $C_5^{2n} \times K_2$. Taking into account $\alpha(C_5^{2n} \times K_2) = \alpha(C_5^{2n}) = 5^n$, we get $|S| \leq 2 \cdot 5^n$, which contradicts to the condition of the Theorem.

The Theorem is thus proved.

Thus, it makes sense to find maximal subgraph of $C_5^{2n} \times K_2$, which is an independent extension of C_5 (which necessarily contains an odd cycle).

Let's prove that $\alpha(C_5^3)=10$ making use of this method.

Assume $\alpha(C_5^3)>10$, in that case there exists a subgraph S of $C_5^2\times K_2$, which is an independent extension of C_5 with cardinal number greater than 10. The fact that cardinal number of S is greater than 10 implies that there exist 2 adjacent rows in C_5^2 (denote L_2 the subgraph induced by the vertices of the rows), so that S has 5 vertices in $L_2\times K_2$ (note that it can't have more than 5 vertices in $L_2\times K_2$, since S is an independent extension of C_5). Analogously there exist 2 adjacent columns (denote C_2) in C_5^2 satisfying the mentioned conditions. Taking into account that S is an independent extension of C_5 in $C_5^2\times K_2$ graph, we get (without loss of generality) the following 4 possible ways vertices of S can be distributed into above mentioned 2 columns and rows (for simplicity only C_5^2 is pictured instead of $C_5^2\times K_2$) (Fig. 3).

x			x	x
	x	x		
x			?	
x		?	?	
	x	?		

a

x				x
	x	x	x	
x				
x		?	?	
	x	?	?	

b

x			x	x
	x	x		
x			?	?
	x		?	?
	x			

c

x				x
	x	x	x	
x				?
	x		?	?
	x		?	

d

Fig. 3.

The remaining vertices of S can be placed only in the cells with question marks. It can be checked that no 3 vertices of S can be placed in those cells, so that S remains an independent extension of C_5. Therefore, $\alpha(C_5^3)=10$.

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