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APPROXIMATION BY POISED SETS OF NODES

G. S. AVAGYAN, L. R. RAFAYELYAN*

Chair of Numerical Analysis and Mathematical Modeling YSU, Armenia

In the present paper it has been shown that nodes of any finite set $X \subset \mathbb{R}^d$ can be made independent by arbitrarily small perturbation, in other words, the set X can be approximated by sets of independent nodes. In the case of $\#X = \dim \prod_n^d$ the set X can be approximated by sets of poised nodes.

Keywords: Lagrange interpolation, independent points, poised sets.

Introduction. Let $\prod_n^d$ be the space of all polynomials in d variables of total degree $\le n$.

Definition 1. The interpolation problem with a set of nodes $X_s = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1,2,...,s\}$ and $\prod_n^d$ is called poised (unisolvent), if for any data $\{c_1,...,c_s\}$ there is a unique polynomial $p \in \prod_n^d$ such that $p(\mathbf{x}_i) = c_i$, $i = 1,...,s$.

This gives us a system of s linear equations with $\binom{n+d}{d}$ unknowns – the coefficients of polynomial. The poisedness means that this system has a unique solution for any right-hand side values. This implies the following necessary condition of poisedness: $s = N := N_n^d = \dim \prod_n^d = \binom{n+d}{d}$.

Thus, we consider poisedness, only if this condition holds. In the latter case we have: the linear system is unisolvent for arbitrary right-hand side values c_i, $i = 1,2,...,N$, iff the corresponding homogeneous system has no solution except the trivial one. So, we can state the following

Proposition 1. The interpolation problem with a set of nodes $X = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1,2,...,N\}$ and $\prod_n^d$ is poised, iff $p \in \prod_n^d$, $p(\mathbf{x}_i) = 0$, $i = 1,...,N \Rightarrow p = 0$.

Consider a set of nodes $X_s = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1,2,...,s\}$. A polynomial p is called a fundamental polynomial for $A := \mathbf{x}_k \in X_s$ and $\prod_n^d$, if $p \in \prod_n^d$, $p(\mathbf{x}_k) = 1$, $p(\mathbf{x}_i) = 0$, $1 \le i \le s$, $i \ne k$.

* E-mail: levon.rafayelyan@gmail.com

We denote fundamental polynomial of $A := \boldsymbol{x}_k \in X_s$ by $p_k^* := P_{\{A, X_s\}}^*$.

Definition 2. The set of nodes $X_s = \{\boldsymbol{x}_i \in \mathbb{R}^d : i = 1,2,...,s\}$ is called $\prod_n^d$-independent, if all fundamental polynomials $p_i^* \in \prod_n^d$, $i = 1,...,s$, exist. Otherwise, X_s is called dependent for $\prod_n^d$, i.e. X_s is dependent for $\prod_n^d$, iff

$$p \in \prod_n^d, \quad p(\boldsymbol{x}_i) = 0, \ 1 \le i \le s, \ i \ne i_0 \Rightarrow p(\boldsymbol{x}_{i_0}) = 0.$$

The following statements are evident.

Proposition 2. Assume that a set of nodes X is $\prod_n^d$-independent. Then:

1. Any nonempty subset $Y \subset X$ is $\prod_n^d$-independent.
2. $\#X \le N = \dim \prod_n^d$, and if $\#X = N$, then set X is $\prod_n^d$-poised.

In the sequel we use the following notation. Let us enumerate the monomials in d variables and denote them by $\varphi_j(\boldsymbol{x})$, $\boldsymbol{x} = (x_1,...,x_d)$, $j = 1,2,...$ Numbering is performed such that monomials of the same degree are numbered in arbitrary order, e.g. lexicographical, and the monomials of different degrees are numbered in increasing order. In particular, $\varphi_1(\boldsymbol{x}) = 1$. For example, if the monomials of the same degree are numbered in lexicographical order, then

$$\varphi_2(\boldsymbol{x}) = x_1,..., \quad \varphi_{d+1}(\boldsymbol{x}) = x_d, \ \varphi_{d+2}(\boldsymbol{x}) = x_1^2, \ \varphi_{d+3}(\boldsymbol{x}) = x_1 x_2,...$$

Thus, all monomials in d variables of degree not exceeding n are $\varphi_j(\boldsymbol{x})$, $j = 1,2,...,N$.

Now let $X = \{\boldsymbol{x}_i \in \mathbb{R}^d : i = 1,2,...,s\}$ be a set of nodes with $s \le N$. The Vandermonde matrix $V_n(X) = [\varphi_1(\boldsymbol{x}_i), \varphi_2(\boldsymbol{x}_i),...,\varphi_N(\boldsymbol{x}_i)]_{i=1}^s$ is an $s \times N$-matrix with $s = \#X$ rows and $N = \dim \prod_n^d = \binom{n+d}{d}$ columns. It is easy to see that $V_n(X)$ is of full rank, i.e. $\operatorname{rank} V_n(X) = \#X$, iff the points of X are $\prod_n^d$-independent.

Approximation by Poised Sets of Nodes. In the sequel we use the following [1]

Lemma. Let $A = [a_{i1}, a_{i2},...,a_{is}]_{i=1}^r$, $r \ge 1$ be a matrix of rank r and $\Omega \subset \mathbb{R}^d$ be a set such that for any $k = 1,2,...$ it contains points not lying on hyper-surface of order k. Then, there exists a point $\boldsymbol{x}^{(1)} \in \Omega$ such that the rank of matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rs} \\ \varphi_1(\boldsymbol{x}^{(1)}) & \varphi_2(\boldsymbol{x}^{(1)}) & \cdots & \varphi_s(\boldsymbol{x}^{(1)}) \end{pmatrix} \text{ is } r+1.$$

Let $\varepsilon > 0$, $\boldsymbol{x}_0 \in \mathbb{R}^d$ and $B(\boldsymbol{x}_0, \varepsilon)$ be the open ball with center $\boldsymbol{x}_0$ and radius ε: $B(\boldsymbol{x}_0, \varepsilon) := \{\boldsymbol{x}_0 \in \mathbb{R}^d : \| \boldsymbol{x}_0 - \boldsymbol{x} \| < \varepsilon\}$.

From Lemma we get the following

Proposition 3. Suppose $X = \{\boldsymbol{x}_i \in \mathbb{R}^d : i = 1,2,...,s\}$ is $\prod_n^d$-independent set of points. If $s < N$, then for any $\varepsilon > 0$ and $\boldsymbol{x}_0 \in \mathbb{R}^d$ there exists a point $\boldsymbol{x}' \in B(\boldsymbol{x}_0, \varepsilon)$ such that the set $X \bigcup \{\boldsymbol{x}'\}$ is $\prod_n^d$-independent.

Proof. Consider the Vandermonde matrix $V_n(X)$. From $\prod_n^d$-independence of X we have that $\operatorname{rank} V_n(X) = \# X = s$. Since any hypersurface cannot contain a point with its neighbourhood in $\mathbb{R}^d$, then the condition of Lemma holds for $\Omega = B(\boldsymbol{x}_0, \varepsilon)$. Thus, there exists a point $\boldsymbol{x}' \in \Omega$ such that $\operatorname{rank} V_n(X \bigcup \{\boldsymbol{x}'\}) = s + 1$. So, the set $X \bigcup \{\boldsymbol{x}'\}$ is $\prod_n^d$-independent.

The following theorem shows that any finite set of nodes $X \subset \mathbb{R}^d$ can be approximated by a set of independent nodes.

Theorem. Let $X_s = \{\boldsymbol{x}_i \in \mathbb{R}^d : i = 1,2,...,s\}$, $s \le N$. For arbitrary $\varepsilon > 0$ there exists a $\prod_n^d$-independent set $Y_s = \{\boldsymbol{y}_i \in \mathbb{R}^d : i = 1,2,...,s\}$ such that $\| \boldsymbol{x}_i - \boldsymbol{y}_i \| < \varepsilon$, $i = 1,2,...,s$.

Proof. We shall prove the statement using the method of mathematical induction on s. Clearly, the statement holds for $s = 1$. Suppose the statement holds for all *k*, $k < s$. Consider the set $X_{s-1} = \{\boldsymbol{x}_i \in \mathbb{R}^d : i = 1,2,...,s-1\}$. The induction hypothesis implies that for arbitrary $\varepsilon > 0$ there exists a $\prod_n^d$-independent set $Y_{s-1} = \{\boldsymbol{y}_i \in \mathbb{R}^d : i = 1,2,...,s-1\}$ such that $\| \boldsymbol{x}_i - \boldsymbol{y}_i \| < \varepsilon$, $i = 1,2,...,s-1$. By the Proposition 3 there exists a point $\boldsymbol{y}_s \in B(\boldsymbol{x}_s, \varepsilon)$ such that the set $Y_s = Y_{s-1} \bigcup \{\boldsymbol{y}_s\}$ is $\prod_n^d$-independent.

Now we readily get the following

Corollary. Consider a set of points $X = \left\{\boldsymbol{x}_i \in \mathbb{R}^d : i = 1,2,...,N\right\}$. For arbitrary $\varepsilon > 0$ there exists a $\prod_n^d$-poised set $Y = \left\{\boldsymbol{y}_i \in \mathbb{R}^d : i = 1,2,...,N\right\}$ such that $\| \boldsymbol{x}_i - \boldsymbol{y}_i \| < \varepsilon$, $i = 1,2,...,N$.

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