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MINIMUM LINEAR ARRANGEMENT OF THE TRANSITIVE ORIENTED, BIPARTITE GRAPHS

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We study the minimum linear arrangement of the graphs (MINLA) on transitive oriented graphs. We prove that MINLA of transitive oriented graphs is NP-complete.

***Keywords*:** linear arrangement, transitive oriented graphs, NP-completeness.

Introduction. It was proved that this problem is NP-complete in general both for oriented graphs and for simple graphs, also it remains NP-complete for simple bipartite graphs [1]. For more specific classes, like trees [2], rooted trees [3], hypercube [4], rectangular lattice [5], it was proved that this problem is solvable in polynomial time. Also, it was proved that the problem is NP-complete for interval graphs, permutation graphs and undirected comparability graphs [6]. But the NP-completeness of the MINLA for the class of transitive oriented graphs was an open problem up to now.

We will reduce the problem of minimum linear arrangement of the vertices of an undirected graph (MINLAR) to minimum linear arrangement of the transitive oriented, bipartite graph (MINLARTB). Recall that the MINLAR is formulated as follows [1].

Note: it is important to distinguish MINLA of transitive oriented and MINLA of transitive orientable graphs. NP-completeness of MINLA of transitive orientable graphs is trivially followed from NP-completeness of MINLA of bipartite graphs [1].

Problem: MINLAR.

Condition: given an undirected graph $G'(V', E')$ and an integer C'.

Question: does there exist an one to one mapping $f': V' \to \{1,\dots,|V'|\}$ satisfying

$$\sum_{(u,v)\in E'} |f'(u)-f'(v)| \le C'.$$

Problem: MINLARTB.

Condition: given a transitive oriented, bipartite graph $G(V, E)$ and an integer C.

Question: does there exist an one to one mapping $f: V \to \{1,\dots,|V|\}$ such that the following two conditions are satisfied:

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$$\forall (u,v) \in E \ f(u) < f(v), \tag{1}$$

$$\sum_{(u,v)\in E} \left(f(u) - f(v) \right) \le C . \tag{2}$$

The Complexity of the MINLA Problem on Transitive Oriented, Bipartite Graphs.

Theorem. MINLAR $\propto$ MINLARTB.

Proof: Given a MINLAR problem defined by $G'(V', E')$ and C', we define a MINLARTB problem as follows:

$$V = V' \bigcup \left\{ u_{ik} \left| 1 \le i \le n, \ 1 \le k \le n^4 \right. \right\} \bigcup \left\{ v_{ik} \left| \exists < v_i, v_j > \in E' \right. \right\},$$

$$E = \left\{ < u_{ik}, v_i > \left| 1 \le i \le n, \ 1 \le k \le n^4 \right. \right\} \bigcup \left\{ < v_{ij}, v_j >, \ < v_{ij}, v_j > \left| < v_i, v_j > \in E' \right. \right\},$$

$$C = \left(n^4 + 1 \right) C' + \frac{n^5 \left(n^4 + 1 \right)}{2} + n^4 \, | E' | + 2 \, | E' |^2,$$

where $n = |V'|$.

First assume that f' is an one to one function from V' onto $\{1,...,n\}$ satisfying

$$\sum_{(u,v)\in E'} \left| f'(u) - f'(u) \right| \le C'.$$

Introduce an arrangement function f for $G(V, E)$ satisfying the following 3 conditions:

a) if $f'(v_i) < f'(v_j)$, then $f(v_i) < f(v_j)$;

b) $f(u_{ik}) = f(v_i) - k - p\,(\text{vi})$ for every $1 \le i \le n$ and $1 \le k \le n^4$, where $p(v_i)$ is the number of vertices that adjacent to v_i and the value of the f' on this vertices is greater then $f'(v_i)$;

c) if $f'(v_i) < f'(v_j)$ and $< v_i, v_j > \in E'$, then $f(u_{i1}) < f(v_{ij}) < f(v_i)$.

Clearly, there exists *f* satisfying all these conditions and (1). Now we want to show that condition (2) is satisfied too. The length of edge $< v_{ik}, v_i >$ is

$$f(v_i) - f(u_{ik}) = k + p(v_i).$$

Thus, the total length of all these n^4 edges adjacent to v_i is

$$\sum_{k=1}^{n^4} \left(k + p(v_i) \right) = n^4 p(v_i) + n^4 \left(n^4 + 1 \right) / 2,$$

and the total length over all *i* is

$$\sum_{i=1}^{n} \left(n^4 p(v_i) + \frac{n^4 \left(n^4 + 1 \right)}{2} \right) = n^4 \, | E' | + \frac{n^5 \left(n^4 + 1 \right)}{2}.$$

For each edge $< v_i, v_j > \in E'$ we have now two edges $< v_{ij}, v_i >, < v_{ij}, v_j >$.

Clearly, $f(v_i) - f(v_{ij}) \le |E'|$ and

$$f(v_j)-f(v_{ij})\le(n^4+1)\big(f'(v_j)-f'(v_i)\big)+|E'|.$$

Using this, we can estimate the total length of arrangement f.

$$\sum_{(u,v)\in E}\big(f(v)-f(u)\big)\le\sum_{(u,v)}\Big((n^4+1)\big(f'(v)-f'(u)\big)\Big)+2|E'|)+n^4|E'|+$$

$$+n^5(n^4+1)/2=(n^4+1)\sum_{(u,v)\in E}|f'(u)-f'(v)|+\sum_{(u,v)\in E}2|E'|^2+n^4|E'|+$$

$$+n^5(n^4+1)/2=(n^4+1)C'+2|E'|^2+n^4|E'|+n^5(n^4+1)/2=C.$$

Thus,

$$\sum_{(u,v)\in E}\big(f(v)-f(u)\big)\le C$$

as required.

Next, assume f is an arrangement function of $G(V,E)$ satisfying (1) and (2).

Let us show that there exists an arrangement function f^* satisfying (1) and (2), and in addition, has the following property:
if $\exists i,j$ such that $f^*(v_i)<f^*(v_j)$, then $f^*(v_i)<f^*(u_{jk})<f^*(v_j)$, $k=1,\ldots,n^4$.(*)

We derive f^* from f by successive circular shifts of consecutive sections of vertices without violating (1) and (2).

If the property above does not satisfied, then $\exists v_i,v_j$ and u_{jp} such that $f(u_{jp})<f(v_i)<f(v_j)$ and $f(v_i)-f(u_{jp})\to\min$. We circularly shift all vertices from the interval $\left[f(u_{jp}),f(v_i)\right]$ by one step to the left.

Clearly, the condition (1) is preserved. Let us show that the total length of the edges decreases, thus, (2) remains valid. Assume there are originally q vertices of type u_{jk} in the interval $\left[f(u_{jp}),f(v_i)\right]$.

In this case the remaining n^4-q vertices of type u_{jk} stay on the left hand side of u_{jp}. The shift shortens the edge $<v_{jp},v_j>$ by at least $q+1$. In addition, each of the n^4-q edges $<v_{ik},v_i>$ is shortened by one, so, we have reduced the length by n^4-q. So far, we have reduced the total length by at least n^4+1. The edges $<s,t>$, which may have lengthened by the shift, have the following property:

$$f(u_{jp})<f(s)<f(v_i)\ \text{and}\ f(v_i)<f(t),$$

and since we choose the vertices v_i,v_j and v_{jp} such that $f(v_i)-f(u_{jp})\to\min$, then the type of the vertex s is v_{jp}. However, the total increase of the edges $<v_{jp},t>$ is bounded by $2|E'|$. Since

$$2|E'|\le\frac{2n(n-1)}{2}=n(n-1)<n^4+1,$$

the total length of the edges is decreased. We will repeat this process until (*) is satisfied. Further, let us construct from f^* another f^{**} satisfying

$$\forall i,j\, f^{**}(v_{ij}) < f^{**}(v_i) < f^{**}(v_j) \text{ and } \max_k f^{**}(u_{ik}) < f^{**}(v_{ij}). \qquad (**)$$

If the property above does not satisfied, then $\exists v_i, v_j$ and v_{ij} such that $f^*(v_{ij}) < f^*(u_{ik_l})$, $l = 1,\ldots,q$, $q > 0$. Assume also that there are originally p vertices of type v_{rh} in the interval $\left[f^*(v_{ij}), f^*(v_i)-1\right]$. In this case we circularly shift all vertices from $\left[f^*(v_{ij}), f^*(v_i)-1\right]$ by one step to the left. After this action length of every edge adjacent v_{ij} decreases by $p+q$, and length of every edge starting from u_{ik_l} or v_{rh} increases by 1. Since we have 2 edges for v_{ij}, q edges for u_{ik_l} and $2p$ edges for all v_{rh}, the total length decreases by $2(p+q)$ and increases by $2p+q$. So, we have shortened the total length by $q > 0$. We will repeat this process while (**) is not satisfied. Then we will have an arrangement function f^{**} satisfying (**), and preserving conditions (1) and (2).

Let us construct an arrangement function f' for G', induced by f^{**} by the following way.

We take vertices $v_1,\ldots,v_n$ by their relative positions in f^{**} and enumerate them from 1 to *n*, and set $f'(v_i)$ to be equal to the number that has been associated to v_i in new numbering.

Then $\sum\limits_{(u,v)\in E} \left|f^{**}(v) - f^{**}(u)\right| \le C$, and since f^{**} satisfies conditions a), b) and c), we obtain

$$\sum_{(u,v)\in E'} \left(f^{**}(v_i) - f^{**}(v_{ij}) + f^{**}(v_i) - f^{**}(v_{ij})\right) + n^4|E'| + \frac{n^5(n^4+1)}{2} =$$
$$= \sum_{(u,v)\in E} \left(f^{**}(v) - f^{**}(u)\right).$$

Let us now estimate the following expression for every edge $(v_i, v_j) \in E'$:

$$\left(f^{**}(v_i) - f^{**}(v_{ij}) + f^{**}(v_i) - f^{**}(v_{ij})\right).$$

Clearly,

$$f^{**}(v_j) - f^{**}(v_{ij}) \ge (n^4+1)\left|f'(v_j) - f'(v_i)\right| \text{ and } f^{**}(v_j) - f^{**}(v_{ij}) > 0.$$

Therefore,

$$(n^4+1)\sum_{(u,v)\in E'} \left|f'(u) - f'(v)\right| \le \sum_{(u,v)\in E'} \left(f^{**}(v_i) - f^{**}(v_{ij}) + f^{**}(v_i) - f^{**}(v_{ij})\right)$$

and

$$\left(n^4+1\right)\sum_{(u,v)\in E'}\left|f'(u)-f'(v)\right|+n^4\left|E'\right|+\frac{n^5\left(n^4+1\right)}{2}\le$$

$$\le\sum_{\left(u_i,v_j\right)\in E'}\left(f^{**}\left(v_i\right)-f^{**}\left(v_{ij}\right)+f^{**}\left(v_j\right)-f^{**}\left(v_{ij}\right)\right)+\frac{n^5\left(n^4+1\right)}{2}+n^4\left|E'\right|=$$

$$=\sum_{(u,v)\in E'}\left(f^{**}(v)-f^{**}(u)\right)\le C=\left(n^4+1\right)C'+\frac{n^5\left(n^4+1\right)}{2}+n^4\left|E'\right|+2\left|E'\right|^2.$$

Thus,

$$\sum_{(u,v)\in E'}|f'(u)-f'(v)|\le C'+\frac{2\left|E'\right|^2}{n^4+1}$$

and since

$$\sum_{(u,v)\in E'}\left|f'(u)-f'(v)\right|\in Z_+,\qquad C'\in Z_+\ \text{ and }\ \frac{2\left|E'\right|^2}{n^4+1}<1,$$

then

$$\sum_{(u,v)\in E'}\left|f'(u)-f'(v)\right|\le C'.$$

We prove that MINLARTB graphs are NP-complete.

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REFERENCES

1. **Even S., Shiloah Y.** NP-Completeness of Several Arrangement Problems. Report 43. Dept. of Computer Science. Israel, Haifa: Technion, 1975.
2. **Chung F.R.K.** Labelings of Graphs. San Diego: Academic Press, 1988, p. 151–168.
3. **Adolphson D. and Hu T.C.** // SIAM J. Appl. Mathem., 1973, v. 25, № 3, p. 403–423.
4. **Harper L.H.** // J. SIAM, 1964, v. 12, № 1, p. 131–135.
5. **Muradyan D.O. and Piliposyan T.E.** // Izv. Akad. Nauk Arm. SSR, 1980, v. 1, № 70, p. 21–27 (in Russian).
6. **Cohen J., Fomin F.** "Optimal Linear Arrangement of Interval Graphs". 31st International Symposium on Mathematical Foundations of Computer Science, 2006.