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# LEAST SQUARES DATA FITTING WITH QUADRATIC BEZIER CURVES

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In the present paper a numerical algorithm to construct quadratic Bezier curves for data fitting by least squares method is developed. The problem is solved by constructing so-called minimizing sequence of control points.

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**1. Introduction.** The least squares approximation is the most popular tool for curve fitting of measured data. To be more exact, it is a process of constructing a curve that minimizes the sum of squared residuals between the original data and the values given by the curve (see, e.q., [1]).

Different functions can be used for curve fitting, such as logarithmic and exponential functions, algebraic polynomials, splines, etc. [2]. In the present paper we consider the problem of least squares approximation by quadratic Bezier curves. Note that Bezier curves have many applications in image processing, pattern control and computer graphics [3].

**2. Least Squares Fitting Problem.** Consider a plane with Cartesian coordinates $(x,y)$. The distance between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is defined in a natural way, that is $|A-B| \equiv \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

Given points $P_0(x_0,y_0)$, $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, the quadratic Bezier curve is the path traced by the function [3]

$$B(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2, \quad t \in [0,1]. \tag{2.1}$$

The points $P_0$, $P_1$, $P_2$ are called *control points* of the Bezier curve: $P_0$ is the starting point and $P_2$ is the endpoint.

Let us be given a set of data points

$$q_i(\xi_i,\eta_i), \quad i = 1,2,...,N, \tag{2.2}$$

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where $N > 2$. Suppose that for each point $q_i$ a value $t_i \in [0,1]$ is assigned. From now on $t_i$ will be referred to as a *knot*. Thus, we have a set of knots $T = (t_1, t_2, \ldots, t_N)$ associated with data (2.2). The following notation will be used:

$$E(P_0, P_1, P_2; T) \equiv \sum_{i=1}^{N} |B(t_i) - q_i|^2 . \tag{2.3}$$

The problem of data fitting by Bezier curve (2.1) in the least squares sense is formulated as follows: find control points $P_0, P_1, P_2$, for which the sum (2.3) of square deviations $E(P_0, P_1, P_2; T)$ assumes its minimal value.

In the solution of the proposed problem, the shape of the curve is influenced by the choice of the knots $t_i$, $i = 1, 2, \ldots, N$, corresponding to data points (2.2). One can use different algorithms to determine the knots, depending on the aim pursued [4,5]. It should be noted that the algorithm of choosing this knots is one of the key points in data fitting problem.

Here we consider the following version of data fitting by least squares method. For each data point $q_i$, $1 \le i \le N$, we assign the knot $t_i^d$, which minimizes the distance from that point to the desired curve, i.e.

$$|B(t_i^d) - q_i| = \min_{0 \le t \le 1} |B(t) - q_i| . \tag{2.4}$$

Let $T^d = (t_1^d, t_2^d, \ldots, t_N^d)$. Our main problem is the following: find control points $P_0, P_1, P_2$, such that the sum

$$E(P_0, P_1, P_2; T^d) = \sum_{i=1}^{N} |B(t_i^d) - q_i|^2 \tag{2.5}$$

assumes its minimal value.

Obviously, the knots $t_i^d$, $1 \le i \le N$, depend on the control points $P_0, P_1$ and $P_2$. The relationship is nonlinear and is given implicitly (see (2.4)), so we encounter with sufficiently complex kind of nonlinear optimization. Therefore, below in the paper we develop another approach and instead of the posed problem (2.5) solve a related problem of constructing a *minimizing sequence* of control points.

**3. Determining the Distance from a Point to a Quadratic Bezier Curve.** When constructing an algorithm of curve fitting within the formulated problem (2.5), a computational procedure of determining the distance from a point to the considered curve must be developed. It should be noted that this procedure is an important component part of many algorithms in curve fitting problems.

Suppose a quadratic Bezier curve (2.1) and a point $q(\xi, \eta)$ are given. Define a function

$$f(t) \equiv |B(t) - q|^2 . \tag{3.1}$$

Consider the following problem.

*Problem 1.* Find a knot $t^* \in [0,1]$ such that

$$f(t^*) = \min_{0 \le t \le 1} f(t) . \tag{3.2}$$

As a matter of fact, the knot $t^*$ minimizes the distance between the point $q$ and the curve $B(t)$, that is $B(t^*)$ is the point on the curve $B(t)$ closest to $q$.

By direct calculation, from (3.1) we find

$$f(t) = d_4 t^4 + 4d_3 t^3 + 2d_2 t^2 + 4d_1 t + d_0\,, \tag{3.3}$$

where

$$\begin{aligned} d_4 &\equiv (x_0 - 2x_1 + x_2)^2 + (y_0 - 2y_1 + y_2)^2\,, \\ d_3 &\equiv (x_0 - 2x_1 + x_2)(x_1 - x_0) + (y_0 - 2y_1 + y_2)(y_1 - y_0)\,, \\ d_2 &\equiv (x_0 - 2x_1 + x_2)(x_0 - \xi) + (y_0 - 2y_1 + y_2)(y_0 - \eta) + \\ &\quad + 2(x_1 - x_0)^2 + 2(y_1 - y_0)^2\,, \\ d_1 &\equiv (x_1 - x_0)(x_0 - \xi) + (y_1 - y_0)(y_0 - \eta)\,, \\ d_0 &\equiv (x_0 - \xi)^2 + (y_0 - \eta)^2\,. \end{aligned} \tag{3.4}$$

To solve the Problem 1, we have to find zeros of the equation $f'(t) = 0$. As follows from (3.3), $f'(t) = P(t)$, where

$$P(t) \equiv d_4 t^3 + 3d_3 t^2 + d_2 t + d_1\,. \tag{3.5}$$

The roots of the polynomial $P(t)$ can be found by applying well developed numerical algorithms (see, e.q., [6, 7]). Thus, we get the following procedure to determine the distance from a point to the quadratic Bezier curve.

Procedure **BezDist** $[P_0, P_1, P_2, q \Rightarrow t^*, D^*]$

1. Input $P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2), q(\xi, \eta)$.
2. Compute $d_4, d_3, d_2, d_1, d_0$ by formulae (3.4).
3. Find a set $\{t_\mu\}$ of real roots in [0, 1] of the polynomial $P(t)$ (see (3.5)). $P_2(x_2, y_2)$.
4. Choose the knot $t^*$ from the set $\{\{t_\mu\}, 0, 1\}$ such that $f(t^*) = \min\{\{f(t_\mu)\}, f(0), f(1)\}$.

END

**4. Constructing a Minimizing Sequence of Control Points.** Suppose we have a sequence of triples $(P_0^{(k)}, P_1^{(k)}, P_2^{(k)})$, $k = 0, 1, \ldots$, of control points, so, we have a sequence of quadratic Bezier curves

$$B^{(k)}(t) = (1-t)^2 P_0^{(k)} + 2t(1-t)P_1^{(k)} + t^2 P_2^{(k)}\,, \quad k = 0, 1, \ldots \tag{4.1}$$

For each triple $(P_0^{(k)}, P_1^{(k)}, P_2^{(k)})$ we obtain a set of knots $T^{(k)} = (t_1^{(k)}, t_2^{(k)}, \ldots, t_N^{(k)})$ satisfying

$$|B^{(k)}(t_i^{(k)}) - q_i| = \min_{0 \le t \le 1} |B^{(k)}(t) - q_i|\,, \quad i = 1, 2, \ldots, N\,. \tag{4.2}$$

In accordance with (2.3), let

$$E^{(k)} \equiv E(P_0^{(k)}, P_1^{(k)}, P_2^{(k)}; T^{(k)}) = \sum_{i=1}^{N} |B^{(k)}(t_i^{(k)}) - q_i|^2\,, \quad k = 0, 1, \ldots \tag{4.3}$$

*Definition 1.* We say that the triples of control points $(P_0^{(k)},\, P_1^{(k)},\, P_2^{(k)})$, $k = 0, 1, \ldots$, form a minimizing sequence, if

$$E^{(k)} \le E^{(k-1)}, \quad k = 1, 2, \ldots \tag{4.4}$$

Since the sequences $E^{(k)}$ is non-increasing and bounded from below, then there exists $E^* \ge 0$ such that $\lim\limits_{k \to \infty} E^{(k)} = E^*$.

To construct a minimizing sequence of triples of control points consider first the following "weak" problem.

*Problem 2.* For a given set of knots $T=(t_1,t_2,\dots,t_N)$ find control points $P_0(x_0,y_0)$, $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, for which the sum $E(P_0,P_1,P_2;T)$ defined in (2.3) assumes its minimal value.

The problem is solved as follows. We have

$$\begin{aligned} E(P_0,P_1,P_2;T) \;&=\; \sum_{i=1}^{N}\Big\{\big[(1-t_i)^2x_0+2t_i(1-t_i)x_1+t_i^2x_2-\xi_i\big]^2+ \\ &\qquad +\big[(1-t_i)^2y_0+2t_i(1-t_i)y_1+t_i^2y_2-\eta_i\big]^2\Big\}. \end{aligned}$$

At the point of minimum the partial derivatives

$$\frac{\partial E(P_0,P_1,P_2;T)}{\partial x_i},\ \frac{\partial E(P_0,P_1,P_2;T)}{\partial y_i},\quad i=0,1,2,$$

should be equal to zero, we arrive at the systems of linear equations

$$\sum_{j=0}^{2}\alpha_{ij}x_j=\gamma_i\,,\quad \sum_{j=0}^{2}\alpha_{ij}y_j=\delta_i\,,\quad i=0,1,2\,, \tag{4.5}$$

where

$$\begin{aligned} &\alpha_{00}=\sum_{i=1}^{N}(1-t_i)^4, && \alpha_{01}=2\sum_{i=1}^{N}t_i(1-t_i)^3, && \alpha_{02}=\sum_{i=1}^{N}t_i^2(1-t_i)^2,\\ &\alpha_{11}=4\sum_{i=1}^{N}t_i^2(1-t_i)^2, && \alpha_{12}=2\sum_{i=1}^{N}t_i^3(1-t_i), && \alpha_{22}=\sum_{i=1}^{N}t_i^4,\\ &\alpha_{10}=\alpha_{01}\,, && \alpha_{20}=\alpha_{02}\,, && \alpha_{21}=\alpha_{12} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\gamma_0=\sum_{i=1}^{N}(1-t_i)^2\xi_i\,, && \gamma_1=2\sum_{i=1}^{N}t_i(1-t_i)\xi_i\,, && \gamma_2=\sum_{i=1}^{N}t_i^2\xi_i\,,\\ &\delta_0=\sum_{i=1}^{N}(1-t_i)^2\eta_i\,, && \delta_1=2\sum_{i=1}^{N}t_i(1-t_i)\eta_i\,, && \delta_2=\sum_{i=1}^{N}t_i^2\eta_i\,. \end{aligned} \tag{4.7}$$

To prove the solvability of the systems (4.5), let $A=[\alpha_{ij}]_{i,j=0}^{2}$ be the matrix of the systems. Note that $A$ is positive semi-definite, consequently $\det A\geq 0$.

***L e m m a .*** If $T=(t_1,t_2,\dots,t_N)$ contains at least three distinct knots, then the matrix $A$ is positive definite.

*Proof.* Consider a vector $z=[z_0\,z_1\,z_3]^T$. Taking into account (4.6), we find

$$(Az,z)=\sum_{i=1}^{N}\big[(z_2-2z_1+z_0)t_i^2+2(z_1-z_0)t_i+z_0\big]^2.$$

Thus, $(Az,z)\geq 0$ for any $z$. Further, it can be readily proved that when $z\neq 0$, then $(Az,z)>0$. Indeed, consider a polynomial

$$Q(t)\equiv(z_2-2z_1+z_0)t^2+2(z_1-z_0)t+z_0\,.$$

Since $z\neq 0$, then the coefficients of the polynomial $Q(t)$ can not be equal to zero simultaneously. So, we have $(Az,z)=\sum_{i=1}^{N}Q^2(t_i)\,.$

If $(Az, z) = 0$, then $Q(t_i) = 0$, $i = 1, 2, \ldots, N$. But a polynomial of second order can not have more than two distinct roots. $\square$

Thus, the matrix $A$ is positive definite, yielding $\det A > 0$. On the base of above considerations we can write the computational procedure to solve Problem 2 .

Procedure **BezGivenKnots** $[\{q_i\}_{i=1}^N, T \Rightarrow P_0, P_1, P_2]$

1. Input $\{q_i(\xi_i, \eta_i)\}_{i=1}^N$, $T = (t_1, t_2, \ldots, t_N)$.
2. Compute the coefficients and right-hand sides of the systems (4.5) by formulae (4.6) and (4.7).
3. Solve the systems (4.5); get the control points $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$.
4. Output $P_0$, $P_1$, $P_2$.

END

To construct the basic algorithm, besides of the Problem 2, as we will see below, we need to solve a related problem, when the endpoints $P_0$ and $P_2$ are fixed [8].

*Problem 2A.* For a given endpoints $P_0(x_0, y_0)$, $P_2(x_2, y_2)$ and a set of knots $T = (t_1, t_2, \ldots, t_N)$ find a control point $P_1(x_1, y_1)$, minimizing the sum

$$E(P_1; T) \equiv \sum_{i=1}^{N} |B(t_i) - q_i|^2. \tag{4.8}$$

This problem can be readily solved as follows. We have

$$\begin{aligned} E(P_1;T) \;\; = \;\; & \sum_{i=1}^{N} \Big\{ \left[(1-t_i)^2 x_0 + 2t_i(1-t_i)x_1 + t_i^2 x_2 - \xi_i\right]^2 + \\ & \qquad + \left[(1-t_i)^2 y_0 + 2t_i(1-t_i)y_1 + t_i^2 y_2 - \eta_i\right]^2 \Big\}. \end{aligned}$$

Calculating the partial derivatives of $E(P_1; T)$ with respect to $x_1$ and $y_1$, setting them to be equal to zero, we get equations

$$\begin{aligned} & \alpha x_0 + 2\gamma x_1 + \beta x_2 - \delta_x = 0, \quad \alpha y_0 + 2\gamma y_1 + \beta y_2 - \delta_y = 0, \text{where} \\ & \alpha = \sum_{i=1}^{N} t_i(1-t_i)^3, \quad \beta = \sum_{i=1}^{N} t_i^3(1-t_i), \quad \gamma = \sum_{i=1}^{N} t_i^2(1-t_i)^2, \\ & \delta_x = \sum_{i=1}^{N} t_i(1-t_i)\xi_i, \quad \delta_y = \sum_{i=1}^{N} t_i(1-t_i)\eta_i. \end{aligned} \tag{4.9}$$

If $\gamma \neq 0$, we obtain the following formulae for the unknowns $x_1$ and $y_1$:

$$x_1 = \frac{\delta_x - \alpha x_0 - \beta x_2}{2\gamma}, \quad y_1 = \frac{\delta_y - \alpha y_0 - \beta y_2}{2\gamma}. \tag{4.10}$$

Note, that if $\gamma = 0$, then $t_i = 0$ or $t_i = 1$ for all $i = 1, 2, \ldots, N$. In this case $\alpha = \beta = \delta_x = \delta_y = 0$ as well (see (4.9)). The latter means that the point $P_1(x_1, y_1)$ can be taken arbitrary. With the glance to the subsequent constructions we suggest to choose in this case the point $P_1$ as follows:

$$P_1 = \frac{1}{N} \sum_{i=1}^{N} q_i. \tag{4.11}$$

Summarizing, we arrive at the following computational procedure.

Procedure **BezFixEndsGivenKnots** $[P_0, P_2, \{q_i\}_{i=1}^N, T \Rightarrow P_1]$

1. Input $P_0(x_0, y_0)$, $P_2(x_2, y_2)$, $\{q_i(\xi_i, \eta_i)\}_{i=1}^N$, $T = (t_1, t_2, \ldots, t_N)$.
2. Compute $\alpha$, $\beta$, $\gamma$, $\delta_x$, $\delta_y$ by formulae (4.9).
3. If $\gamma \neq 0$ compute $x_1$ and $y_1$ by formulae (4.10); otherwise compute $P_1$ by formula (4.11).
4. Output $P_1(x_1, y_1)$.

END

Now we are able to solve our basic problem.

*Problem 3.* Construct a minimizing sequence of triples of control points $(P_0^{(k)}, P_1^{(k)}, P_2^{(k)})$, $k = 0, 1, \ldots$, and corresponding sequence of quadratic Bezier curves $B^{(k)}(t)$, $k = 0, 1, \ldots$

Let us describe the steps of the computational process.

*Step 0.* Choose an initial triple of control points: $P_0^{(0)}(x_0^{(0)}, y_0^{(0)})$, $P_1^{(0)}(x_1^{(0)}, y_1^{(0)})$ and $P_2^{(0)}(x_2^{(0)}, y_2^{(0)})$. By this we get the initial curve

$$B^{(0)}(t) = (1-t)^2 P_0^{(0)} + 2t(1-t)P_1^{(0)} + t^2 P_2^{(0)}, \quad t \in [0,1]. \tag{4.12}$$

Then for every value of $i$, where $1 \le i \le N$, we determine the corresponding knot $t_i^{(0)} \in [0,1]$ with respect to the curve $B^{(0)}(t)$ by using the procedure

$$\textbf{BezDist}\,[P_0^{(0)}, P_1^{(0)}, P_2^{(0)}, q_i \Rightarrow t_i^{(0)}, D_i^{(0)}].$$

As a result we get a starting set of knots $T^{(0)} = (t_1^{(0)}, t_2^{(0)}, \ldots, t_N^{(0)})$ and corresponding sum of squares of deviations

$$E^{(0)} = \sum_{i=1}^{N} |B^{(0)}(t_i^{(0)}) - q_i|^2 = \sum_{i=1}^{N} D_i^{(0)}. \tag{4.13}$$

*Step k* (for $k \ge 1$). As a result of the previous step, we have control points $P_m^{(k-1)}(x_m^{(k-1)}, y_m^{(k-1)})$, $m = 0, 1, 2$, the corresponding Bezier curve

$$B^{(k-1)}(t) = (1-t)^2 P_0^{(k-1)} + 2t(1-t)P_1^{(k-1)} + t^2 P_2^{(k-1)}, \quad t \in [0,1], \tag{4.14}$$

and the set of knots $T^{(k-1)} = (t_1^{(k-1)}, t_2^{(k-1)}, \ldots, t_N^{(k-1)})$.

Running the procedure **BezGivenKnots** $[\{q_i\}_{i=1}^N, T^{(k-1)} \Rightarrow P_0^{(k)}, P_1^{(k)}, P_2^{(k)}]$, we find the next triple of control points $P_m^{(k)}(x_m^{(k)}, y_m^{(k)})$, $m = 0, 1, 2$. Note that it can happen that the procedure **BezGivenKnots** will fail at some step (this will occur when the determinant of systems (4.5) is equal to zero). In this case we set $P_0^{(k)} = P_0^{(k-1)}$, $P_2^{(k)} = P_2^{(k-1)}$ and then run the procedure

$$\textbf{BezFixEndsGivenKnots}\,[P_0^{(k)}, P_2^{(k)}, \{q_i\}_{i=1}^N, T^{(k-1)} \Rightarrow P_1^{(k)}].$$

So, we obtain Bezier curve

$$B^{(k)}(t) = (1-t)^2 P_0^{(k)} + 2t(1-t)P_1^{(k)} + t^2 P_2^{(k)}, \quad t \in [0,1]. \tag{4.15}$$

Further, for every $i$, $1 \le i \le N$, we determine the corresponding knot $t_i^{(k)} \in [0,1]$ with respect to the curve $B^{(k)}(t)$ using the procedure

$$\textbf{BezDist}\,[P_0^{(k)}, P_1^{(k)}, P_2^{(k)}, q_i \Rightarrow t_i^{(k)}, D_i^{(k)}].$$

Thus, we get a new set of knots $T^{(k)} = (t_1^{(k)}, t_2^{(k)}, \ldots, t_N^{(k)})$ and the sum of squares of deviations

$$E^{(k)} = \sum_{i=1}^{N} |B^{(k)}(t_i^{(k)}) - q_i|^2 = \sum_{i=1}^{N} D_i^{(k)}\,. \qquad (4.16)$$

Summarizing, we obtain the following computational process.

Procedure **BezMinSeq** $[\{q_i\}_{i=1}^N \Rightarrow P_0^{(k)}, P_1^{(k)}, P_2^{(k)}, B^{(k)}(t), E^{(k)}]$

1. Input $\{q_i(\xi_i, \eta_i)\}_{i=1}^N$.
2. Preprocessing:
   - 2.1. set an initial triple of control points $P_0^{(0)}, P_1^{(0)}, P_2^{(0)}$;
   - 2.2. get the curve $B^{(0)}(t)$ (see (4.12));
   - 2.3. for the values $i = 1, 2, \ldots, N$ do:
     run **BezDist** $[P_0^{(0)}, P_1^{(0)}, P_2^{(0)}, q_i \Rightarrow t_i^{(0)}, D_i^{(0)}]$;
     get $T^{(0)} = (t_1^{(0)}, t_2^{(0)}, \ldots, t_N^{(0)})$;
   - 2.4. compute $E^{(0)}$ by formula (4.13).
3. For the values $k = 1, 2, \ldots$ do:
   - 3.1. run **BezGivenKnots** $[\{q_i\}_{i=1}^N, T^{(k-1)} \Rightarrow P_0^{(k)}, P_1^{(k)}, P_2^{(k)}]$;
     if the procedure has been failed, set $P_0^{(k)} = P_0^{(k-1)}$, $P_2^{(k)} = P_2^{(k-1)}$
     and run **BezFixEndsGivenKnots** $[P_0^{(k)}, P_2^{(k)}, \{q_i\}_{i=1}^N, T^{(k-1)} \Rightarrow P_1^{(k)}]$;
   - 3.2. get the curve $B^{(k)}(t)$ (see (4.15));
   - 3.3. for the values $i = 1, 2, \ldots, N$ do:
     run **BezDist** $[P_0^{(k)}, P_1^{(k)}, P_2^{(k)}, q_i \Rightarrow t_i^{(k)}, D_i^{(k)}]$;
     get $T^{(k)} = (t_1^{(k)}, t_2^{(k)}, \ldots, t_N^{(k)})$;
   - 3.4. compute $E^{(k)}$ by formula (4.16);
   - 3.5. output $P_0^{(k)}$, $P_1^{(k)}$, $P_2^{(k)}$, $B^{(k)}(t)$, $E^{(k)}$.

END

Note that as a criterion of stopping of the computational process defined by the procedure **BezMinSeq**, we can take the condition $E^{(k-1)} - E^{(k)} < \varepsilon$ with some prescribed $\varepsilon > 0$.

Finally, let us prove that the procedure **BezMinSeq** actually leads to a minimizing sequence.

***Theorem.*** The sequence $E^{(k)}$, computed by the procedure **BezMinSeq**, is non-increasing:

$$E^{(k)} \le E^{(k-1)}, \quad k = 1, 2, \ldots \qquad (4.17)$$

*Proof.* As follows from the principle of computing the knots $t_i^{(k)}$, $i = 1, 2, \dots, N$,

$$\begin{aligned} E^{(k)} &= E(P_0^{(k)}, P_1^{(k)}, P_2^{(k)}; T^{(k)}) = \sum_{i=1}^{N} |B^{(k)}(t_i^{(k)}) - q_i|^2 \leq \\ &\leq \sum_{i=1}^{N} |B^{(k)}(t_i^{(k-1)}) - q_i|^2 = E(P_0^{(k)}, P_1^{(k)}, P_2^{(k)}; T^{(k-1)}) \end{aligned} \tag{4.18}$$

(see (4.2) and (4.3)). Since the control points $P_0^{(k)}, P_1^{(k)}, P_2^{(k)}$ are the solution of the Problem 2 (or the Problem 2A) with given knots $T^{(k-1)}$ (see Step $k$ of the iterative process), then

$$E(P_0^{(k)}, P_1^{(k)}, P_2^{(k)}; T^{(k-1)}) \leq E(P_0^{(k-1)}, P_1^{(k-1)}, P_2^{(k-1)}; T^{(k-1)}) = E^{(k-1)}. \tag{4.19}$$

Thus, from (4.18) and (4.19) we obtain the inequality (4.17). □

**Concluding Remarks.** In this paper we have developed an approach to solve a problem regarding least squares data fitting by quadratic Bezier curves. The solution of the problem has been obtained by constructing a minimizing sequence of control points. Numerical experiments confirm the practical effectiveness of the proposed method. We hope that the idea discussed in the paper can be useful also in data fitting problems with high order Bezier curves.

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