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Mathematics

ON ZIGZAG DE MORGAN FUNCTIONS

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There are five precomplete classes of De Morgan functions, four of them are defined as sets of functions preserving some finitary relations. However, the fifth class – the class of zigzag De Morgan functions, is not defined by relations. In this paper we announce the following result: zigzag De Morgan functions can be defined as functions preserving some finitary relation.

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1. Introduction. It is well known that the free Boolean algebra on n free generators is isomorphic to the Boolean algebra of Boolean functions of n variables. The free bounded distributive lattice on n free generators is isomorphic to the lattice of monotone Boolean functions of n variables. Analogous to these facts we have introduced the concept of De Morgan functions and proved that the free De Morgan algebra on n free generators is isomorphic to the De Morgan algebra of De Morgan functions of n variables [1].

The Post's functional completeness theorem for Boolean functions plays an important role in discrete mathematics [2]. In the paper [3] we have established a functional completeness criterion for De Morgan functions. In this paper we show that zigzag De Morgan functions, which are used in the formulation of the functional completeness theorem, can be defined by a finitary relation.

Definition 1. Let X be a nonempty set and $f : X^n \to X$ be a function and let $\mathcal{R} \subseteq X^k$ be a k-ary relation on X. We say that f preserves the relation $\mathcal{R}$, if for any $x_{ij} \in X, i = 1,\dots,k,\ j = 1,\dots,n$, we have

$$(x_{1j}, x_{2j},\dots,x_{kj}) \in \mathcal{R},\ j = 1,\dots,n \Rightarrow,$$

$$(f(x_{11},x_{12},\ \dots,x_{1n})\ ,\dots, f(x_{k1},x_{k2},\dots,x_{kn})) \in \mathcal{R}. \qquad (1)$$

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For a nonempty set X and a finitary relation $\mathcal{R} \subseteq X^k$ we say that a class $\mathfrak{M}$ of functions (operations) on X is defined by $\mathcal{R}$, if $\mathfrak{M}$ is the set of all functions on X that preserve the relation $\mathcal{R}$.

2. De Morgan Functions. Denote $D = \{0, a, b, 1\}$. Defining $0 + x = x + 0 = x$, $0 \cdot x = x \cdot 0 = 0$ and $1 \cdot x = x \cdot 1 = x$, $1 + x = x + 1 = 1$ and $x + x = x$, $x \cdot x = x$ for all $x \in D$, and $a + b = b + a = 1$, $a \cdot b = b \cdot a = 0$, $\overline{0} = 1$, $\overline{1} = 0$, $\overline{a} = a$, $\overline{b} = b$, we get the De Morgan algebra $\mathbf{4} = (D; \{+, \cdot, \bar{}, 0, 1\})$.

Definition 2. A function $f : D^n \to D$ is called a *De Morgan function*, if it preserves the binary relations $\sigma = \{(0,0), (a,b), (b,a), (1,1)\} \subseteq D^2$ and $\rho = \{(b,b), (b,0), (b,1), (b,a), (0,0), (0,a), (1,1), (1,a), (a,a)\} \subseteq D^2$ [1,3].

Denote the set of all De Morgan functions of n variables by $\mathcal{D}_n$. For two functions $f, g : D^n \to D$ define $f + g$, $f \cdot g$ and $\overline{f}$ in the standard way. The set $\mathcal{D}_n$ is closed under operations $+, \cdot, \bar{}$, and, therefore, we get an algebra $\mathfrak{D}_n = (\mathcal{D}_n, \{+, \cdot, \bar{}, 0, 1\})$ (here 0 and 1 are the constant De Morgan functions), which obviously is a De Morgan algebra.

Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in 2^{\{1,\dots,n\}} \times 2^{\{1,\dots,n\}}$. We say that $a \subseteq b$, if $a_1 \subseteq b_1$ and $a_2 \subseteq b_2$. In this way we get a partially ordered set $2^{\{1,\dots,n\}} \times 2^{\{1,\dots,n\}} (\subseteq)$. For an antichain $S \subseteq 2^{\{1,\dots,n\}} \times 2^{\{1,\dots,n\}}$ define the function $f_S : D^n \to D$ in the following way:

$$f_S(x_1, \dots, x_n) = \sum_{s=(s_1,s_2)\in S} \left(\prod_{i\in s_1} x_i \cdot \prod_{i\in s_2} \overline{x}_i \right). \qquad (2)$$

Notice that f_S does not depend on the order of the elements in the set S [4,5,7].

Note that we set $f_\varnothing = 0$ and $f_{\{(\varnothing,\varnothing)\}} = 1$.

For any De Morgan function f of n variables there exists a unique antichain $S \subseteq 2^{\{1,\dots,n\}} \times 2^{\{1,\dots,n\}}$ such that $f = f_S$ [1,3].

3. On the Functional Completeness Theorem. The concepts of essential and inessential variables, superposition of functions, (functionally) closed and (functionally) complete classes for De Morgan functions are defined analogously to those concepts for Boolean functions [2].

Now let us introduce some important closed classes of De Morgan functions, which will be used in the formulation of the functional completeness theorem.

For $u \in D$ denote by $\mathfrak{T}_u$ the set of all De Morgan functions, which preserve the value u, i.e.

$$f \in \mathfrak{T}_u \Leftrightarrow f(u, u, \dots, u) = u.$$

Obviously, $f \in \mathfrak{T}_u$, if and only if f preserves the unary relation $\{u\}$ and, hence, the classes $\mathfrak{T}_u$ are closed. Moreover, they are clones for all $u \in D$.

From the representation of De Morgan functions in DNF we conclude that $\mathfrak{T}_a$ and $\mathfrak{T}_b$ consist exactly of all nonconstant functions, i.e. $\mathfrak{T}_a = \mathfrak{T}_b$. Denote

$$\mathfrak{T} = \mathfrak{T}_a = \mathfrak{T}_b = \mathfrak{D} \setminus \{0, 1\}.$$

The next class we will define is the class of quasimonotone functions.

Definition 3. A De Morgan function $f : D^n \to D$ is called *quasimonotone,* if $f|_{\{0,1\}^n}$ is a monotone Boolean function, i.e. if for any two n-tuples

$\alpha = (\alpha_1, \ldots, \alpha_n),\ \beta = (\beta_1, \ldots, \beta_n) \in \{0,1\}^n$ the condition $\alpha_i \leq \beta_i,\ i = 1, \ldots, n$, implies $f(\alpha) \leq f(\beta)$ [3].

Denote the set of all quasimonotone De Morgan functions by $\mathfrak{QM}$. Obviously, $f \in \mathfrak{QM}$ if and only if f preserves the order relation $\{(0,0),(0,1),(1,1)\} \subseteq D^2$. Hence, $\mathfrak{QM}$ is a clone.

Now let us give some notations, which will be used in the definition of the next important class of De Morgan functions.

For an n-tuple $\gamma = (\gamma_1, \ldots, \gamma_n) \in D^n$ and for $u \in D$ denote $S_\gamma(u) = \{i : 1 \leq i \leq$ $\leq n,\ \gamma_i = u\}$ and $|\gamma_u| = |S_\gamma(u)|$ (here for a finite set A we denote by $|A|$ its cardinality, i.e. the number of its elements). Now let $S_\gamma(a) = \{i_1, \ldots, i_m\},\ S_\gamma(b) = \{j_1, \ldots, j_k\}$. Also let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \{0,1\}^m,\ \beta = (\beta_1, \ldots, \beta_k) \in \{0,1\}^k$ and $i_1 < i_2 < \ldots <$ $< i_m,\ j_1 < j_2 < \ldots < j_k$. For $i = 1, 2, \ldots, n$ denote

$$\delta_i(a) = \begin{cases} \gamma_i, & \text{if } i \notin S_\gamma(a), \\ \alpha_l, & \text{if } i = i_l \in S_\gamma(a) \end{cases}$$

and

$$\delta_i(b) = \begin{cases} \gamma_i, & \text{if } i \notin S_\gamma(b), \\ \beta_l, & \text{if } i = i_l \in S_\gamma(b). \end{cases}$$

And finally denote $\gamma_a(\alpha) = (\delta_1(a), \ldots, \delta_n(a)),\ \gamma_b(\beta) = (\delta_1(b), \ldots, \delta_n(b))$. In other words, if we move along the components of γ from left to right and replace every component a by corresponding α_i (i.e. i is the number of a's from the beginning of γ to current position including that position too), then we get $\gamma_a(\alpha)$. Analogously $\gamma_b(\beta)$ can be described. So, if we write $\gamma_a(\alpha)$, then we suppose that the number of components of α is equal to the number of a's among the components of γ. Also, if that number is zero, then we mean $\gamma_a(\alpha) = \gamma$. The same is valid for $\gamma_b(\beta)$ too. To avoid complex notations we will write $\gamma_{a,b}(\alpha, \beta)$ instead of $(\gamma_a(\alpha))_b(\beta)$ (i.e. $\gamma_{a,b}(\alpha, \beta)$ is obtained from γ, if we replace all a's by corresponding α_i's and all b's by corresponding β_i's).

Definition 4. A De Morgan function $f : D^n \to D$ is called a *zigzag* De Morgan function, if for every n-tuple $\gamma = (\gamma_1, \ldots, \gamma_n) \in D^n$ with $f(\gamma) \in \{0,1\}$ the following condition holds:

for any $\alpha, \alpha' \in \{0,1\}^{|\gamma_a|},\ \beta, \beta' \in \{0,1\}^{|\gamma_b|}$ with

$$f(\gamma_a(\alpha)) \neq f(\gamma_a(\alpha')),\ f(\gamma_b(\beta)) \neq f(\gamma_b(\beta'))$$

we have

$$f(\gamma_{a,b}(\alpha, \beta)) = f(\gamma_{a,b}(\alpha', \beta)) = f(\gamma_{a,b}(\alpha, \beta')) = f(\gamma_{a,b}(\alpha', \beta')).$$

Denote by $\mathfrak{Z}$ the class of all zigzag De Morgan functions. Thus, we have five closed classes of De Morgan functions, namely $\mathfrak{T}_0, \mathfrak{T}_1, \mathfrak{T}, \mathfrak{QM}, \mathfrak{Z}$. Note that all these classes are clones as they contain the projection functions.

In [3] it is formulated a problem to find out whether zigzag De Morgan functions can be described by some finitary relations on the set D or not. In this paper we show that zigzag De Morgan functions can be defined as functions preserving a finitary relation on the set D.

Observe that none of the defined five classes is contained in any other.

Now we present the functional completeness theorem for De Morgan functions that is the main result of the paper [3].

Theorem 1. *(Functional Completeness Theorem.)* A class $\mathfrak{B}$ of De Morgan functions is functionally complete, if and only if none of the five closed classes $\mathfrak{T}_0, \mathfrak{T}_1, \mathfrak{T}, \mathfrak{QM}, \mathfrak{Z}$ contains $\mathfrak{B}$.

4. Zigzag De Morgan Functions Can be Defined by a Finitary Relation. For a finitary relation $\mathcal{R} \subseteq D^n$ denote by $\mathfrak{D}_{\mathcal{R}}$ the class of those De Morgan functions that preserve the relation $\mathcal{R}$. Let us consider all 4-ary relations on the set D, i.e. all subsets of the set D^4. Denote by $\mathbb{M}$ the system of all classes of De Morgan functions that preserve some 4-ary relation on D, i.e.

$$\mathbb{M} = \{\mathfrak{D}_{\mathcal{R}} | \mathcal{R} \subseteq D^4\}.$$

Note that some of this classes may be empty or coincide with $\mathfrak{D}$.

Lemma 1. If a class $\mathfrak{B}$ of De Morgan functions is not included in any of the classes from $\mathbb{M}$, then $[\mathfrak{B}]$ contains all De Morgan functions of one variable.

This lemma is an analogue of the Post's functional completeness theorem for Boolean functions.

Theorem 2. Zigzag De Morgan functions can be defined by some finitary relation on the set D.

The proofs will be given in [7].

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REFERENCES

1. **Movsisyan Yu.M., Aslanyan V.A.** A Functional Representation of Free De Morgan Algebras. // Proceedings of the YSU, Physical and Mathematical Sciences, 2012, № 3, p. 14–16.
2. **Crama Y., Hammer P.L.** Boolean Functions: Theory, Algorithms and Applications, NY, Cambridge University Press, 2011.
3. **Movsisyan Yu.M., Aslanyan V.A.** A Functional Completeness Theorem for De Morgan Functions. // Discrete Applied Mathematics 2014, v. 162, p 1–16, doi:10.1016/j.dam.2013.08.006.
4. **Movsisyan Yu.M.** Hyperidentities of Boolean Algebras. // Izvestiya RAN: Ser. Mat., 1992, v. 56, p. 654–672. English transl. in Russ. Acad. Sci. Izv. Math., 1993, v. 40, p. 607–622.
5. **Movsisyan Yu.M.** Algebras with Hyperidentities of the Variety of Boolean Algebras. // Izvestiya RAN: Ser. Mat. 1996, v. 60, p. 127–168.
6. **Movsisyan Yu.M., Aslanyan V.A.** Hyperidentities of De Morgan Algebras. // Logic Journal of IGPL, 2012, v. 20, p. 1153–1174, doi:10.1093/jigpal/jzr053.
7. **Aslanyan V.A.** Characterization of Zigzag De Morgan Functions. // Discrete Mathematics, Algorithms and Applications. Submitted.