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Mathematics

ON THE TWO-LEVEL PRECONDITIONING IN LEAST SQUARES METHOD

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In the present paper an approach to construct algebraic two-level preconditioners for the matrices of normal systems arising in data fitting by least squares method with piecewise linear basis functions is proposed. The approach is based on using hierarchical grids with their subdivision into substructures and corresponding partition of the matrices. Estimates for condition numbers of preconditioned matrices are obtained.

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1. Introduction. Least squares method can be considered as a method of data fitting. The problem consists in determining the curve that best describes the relationship between expected and observed sets of data by minimizing the sums of the squares of deviations between observed and expected values (see [1], for example). Mathematically, it can be stated as a problem of finding an approximate solution to an overdetermined system of linear equations.

Suppose a set of points (*data points*) (x_i, y_i), $i = 1, 2, \dots, N$, is given. The least squares problem is formulated as follows: *find a function $f(x)$ that depends on some parameters which minimizes the sum*

$$\sum_{i=1}^{N} [f(x_i) - y_i]^2. \tag{1.1}$$

In the least squares method the approximating function $f(x)$ is often sought as a linear combination of basis functions $\varphi_j(x)$, $j = 1, 2, \dots, n$, namely,

$$f(x) = \sum_{j=1}^{n} c_j \varphi_j(x).$$

Usually $n \ll N$ is taken. As a result, the problem is reduced to a *normal system*

$$A^T A c = A^T y, \tag{1.2}$$

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where

$$A = [\varphi_j(x_i)], \quad i = 1,2,\dots,N, \quad j = 1,2,\dots,n, \tag{1.3}$$

is an $N \times n$ matrix and $c = [c_1\, c_2 \dots c_n]^T$, $y = [y_1\, y_2 \dots y_N]^T$ (for example, [2]). One of the drawbacks of the normal system (1.2) is that it is usually ill-conditioned. Furthermore, the condition number of the matrix A^TA increases with the increase of the number of basis functions [3]. This creates certain difficulties for numerical solution of the normal system (1.2). From now on, the matrix A^TA of the normal system (1.2) will be referred to as *NS-matrix.*

The basis functions $\varphi_j(x)$ can be chosen in different ways. In practice, functions with local supports are convenient, e.g., basis splines. In the paper linear basis splines are considered. The ill-conditionedness of corresponding normal systems is discussed in [4].

In this article we develop a new approach that consists in constructing preconditioners for the matrices of normal systems. The concept of preconditioning and its role in the solution of linear systems can be found in [5], for example.

Algebraic multilevel preconditioning method is one of the most efficient tools for numerical solution of large-scale linear systems arising in finite element approximation of partial differential equations. The main ideas of the method have been proposed in [6, 7]. A multilevel approach supposes a recursive decomposition of the grid until a coarse one, where the condition number of corresponding matrix is rather small and the system of grid equations can be solved easily enough. An important part of getting multilevel preconditioners are so-called *two-level preconditioners*. The construction of that preconditioners is based on a special two-level ordering of the nodes on each grid refinement level and corresponding partitioning of the stiffness matrices. The present work is the first attempt to disseminate multilevel technique for partial differential equations to the case of normal systems.

2. Hierarchical Partition of the Segment and *NS*-Matrices. Let (a,b) be an interval containing all the points of the set $X = \{x_1, x_2, \dots, x_N\}$. First we describe a hierarchical partition of the segment $[a,b]$.

Choose an integer $n_0 \geq 2$ and carry out an *initial partition* of the segment $[a,b]$ using the points

$$t_m^{(0)} = a + (m-1)h_0, \quad m = 1,2,\dots,n_0, \tag{2.1}$$

where $h_0 = \dfrac{(b-a)}{(n_0-1)}$ is the meshsize of the partition. The points (2.1) will be referred to as the *nodes* of the initial partition. They form the *initial grid* $\sigma_0 = \{t_m^{(0)}\}_{m=1}^{n_0}$. With the initial partition we associate the intervals $\Delta_m^{(0)} = (t_m^{(0)}, t_{m+1}^{(0)})$, $m = 1,2,\dots,n_0-1$.

Let us fix some integer $p > 3$ and construct a hierarchical sequence of grids $\sigma_0,\ \sigma_1,\ \dots,\ \sigma_p$, where each successive grid $\sigma_k = \{t_m^{(k)}\}_{m=1}^{n_k}$, $k \geq 1$, with *nodes* $t_m^{(k)}$ is obtained from the previous grid $\sigma_{k-1} = \{t_m^{(k-1)}\}_{m=1}^{n_{k-1}}$ by bisection procedure. We say that the grid σ_k corresponds to the k-th *level* of partitioning of the segment $[a,b]$. For the number of nodes n_k and the meshsize h_k of the grid σ_k the formulae

$$n_k = 2^k(n_0-1)+1, \quad h_k = h_0/2^k \tag{2.2}$$

hold. We introduce the following notation for the intervals associated with k-th level of the hierarchical partition:

$$\Delta_m^{(k)} \equiv (t_m^{(k)}, t_m^{(k)} + h_k), \quad \tilde{\Delta}_m^{(k)} \equiv [t_m^{(k)}, t_m^{(k)} + h_k). \tag{2.3}$$

At the partition level $k \geq 1$ we will distinguish two types of nodes of the grid σ_k, that is, *old* nodes (those of the grid σ_{k-1}) and *new* nodes which appear in a result of bisection procedure. The following ordering of the nodes is used: the old nodes preserve their numbers while the new nodes are numbered in consequtive order from left to right. By construction, for $k \geq 1$ the grid σ_k includes the grid σ_{k-1}. Therefore, the partitioning

$$\sigma_k = \sigma_k^{(1)} \cup \sigma_k^{(2)} \tag{2.4}$$

can be defined in a natural way, where $\sigma_k^{(1)}$ is the subgrid of the old nodes and $\sigma_k^{(2)}$ is the subgrid of the new nodes. If $n_k^{(i)}$, $i = 1, 2$, is the number of nodes in the subgrid $\sigma_k^{(i)}$, then $n_k^{(1)} = n_{k-1}$, $n_k^{(2)} = n_k - n_{k-1}$.

For all levels $k = 0, 1, \ldots, p$, let G_k be the space of grid functions defined on the grid σ_k. These functions can be considered as vectors of lenght n_k. If $k \geq 1$, in accordance with the rule of node numbering, a grid function $v \in G_k$ can be represented in the form

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad v_i \in G_k^{(i)}, \, i = 1, 2, \tag{2.5}$$

where $G_k^{(i)}$ is the space of grid functions defined on the subgrid $\sigma_k^{(i)}$.

Further, let V_k, $0 \leq k \leq p$, be the space of piecewise-linear functions associated with the grid σ_k. To each node $t_m^{(k)} \in \sigma_k$, $1 \leq m \leq n_k$, we put into correspondence a basis function $\varphi_m^{(k)}(x) \in V_k$ such that $\varphi_m^{(k)}(t_j^{(k)}) = \delta_{mj}$, $j = 1, 2, \ldots, n_k$, where δ_{mj} is the Kronecker symbol. The basis just defined is referred to as *nodal hierarchical basis*. Obviously, there is a one-to-one correspondence between piecewise-linear functions $\hat{v} \in V_k$ and grid functions from $v \in G_k$. In this connection we say that the function $\hat{v}$ is the *prolongation* of the grid function v and write: $\hat{v} = prol(v \in G_k : V_k)$.

By considering on each partition level a least squares problem of the type (1.1), we obtain a sequence of *NS*-matrices

$$L^{(k)} \equiv A^{(k)^T} A^{(k)}, \quad k = 0, 1, \ldots, p, \tag{2.6}$$

where, in accordance with (1.3),

$$A^{(k)} = [\varphi_m^{(k)}(x_i)], \quad i = 1, 2, \ldots, N, \quad m = 1, 2, \ldots, n_k, \tag{2.7}$$

is an $N \times n_k$ matrix. It follows from (2.6) and (2.7) that the entries $s_{mq}^{(k)}$ of the symmetric matrix $L^{(k)}$ are computed by the formula

$$s_{mq}^{(k)} = \sum_{i=1}^{N} \varphi_m^{(k)}(x_i) \varphi_q^{(k)}(x_i), \quad m, q = 1, 2, \ldots, n_k. \tag{2.8}$$

Let us make now the following assumption: *the finest grid ω_p is such that each interval $\Delta_m^{(p)}$ (see (2.3)) contains at least one point from the set X.* Under this assumption the matrices $L^{(k)}$ are positive definite [4].

If $k \geq 1$, then in accordance with the partitioning (2.4) of the grid σ_k, the matrix $A^{(k)}$ can be written in the block form $A^{(k)} = [A_1^{(k)} A_2^{(k)}]$ with submatrices $A_1^{(k)}$ and $A_2^{(k)}$ of sizes $N \times n_k^{(1)}$ and $N \times n_k^{(2)}$ correspondingly. In this case the matrix $L^{(k)}$ defined in (2.6) is represented in the block form

$$L^{(k)} = \begin{bmatrix} L_{11}^{(k)} & L_{12}^{(k)} \\ L_{21}^{(k)} & L_{22}^{(k)} \end{bmatrix}, \tag{2.9}$$

where $L_{ij}^{(k)} = A_i^{(k)^T} A_j^{(k)}$, $i, j = 1, 2$. Note that the blocks $L_{11}^{(k)}$ and $L_{22}^{(k)}$ are nonsingular diagonal matrices.

3. Two-Level Preconditioners. The block representation (2.9) of the matrix $L^{(k)}$ can be written as

$$L^{(k)} = \begin{bmatrix} S^{(k)} + L_{12}^{(k)} L_{22}^{(k)^{-1}} L_{21}^{(k)} & L_{12}^{(k)} \\ L_{21}^{(k)} & L_{22}^{(k)} \end{bmatrix}, \tag{3.1}$$

where

$$S^{(k)} \equiv L_{11}^{(k)} - L_{12}^{(k)} L_{22}^{(k)^{-1}} L_{21}^{(k)} \tag{3.2}$$

is the Schur complement (c.f. [8]).

For all values $k = 1, 2, \ldots, p$, taking into account the hierarchical structure of the grids, let us consider the matrix

$$B^{(k)} = \begin{bmatrix} L^{(k-1)} + L_{12}^{(k)} L_{22}^{(k)^{-1}} L_{21}^{(k)} & L_{12}^{(k)} \\ L_{21}^{(k)} & L_{22}^{(k)} \end{bmatrix} \tag{3.3}$$

as a *two-level preconditioner* for the matrix $L^{(k)}$. By construction, the matrix $B^{(k)}$ is positive definite [8].

Now we describe the process of a system solution with matrix $B^{(k)}$, $1 \leq k \leq p$. To that end, let us consider the system

$$B^{(k)} v = g, \tag{3.4}$$

where $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, $v_i, g_i \in G_k^{(i)}$, $i = 1, 2$. By using the block structure (3.3) of the matrix $B^{(k)}$, we obtain the following computational procedure.

Procedure $B^{(k)}/\mathrm{TL}$

1. compute $f_1 = g_1 - L_{12}^{(k)} L_{22}^{(k)^{-1}} g_2$;
2. solve the system $L^{(k-1)} v_1 = f_1$; (3.5)
3. compute $v_2 = L_{22}^{(k)^{-1}} (g_2 - L_{21}^{(k)} v_1)$.

End

Thus, the solution of the system (3.4) is reduced to the solution of the system (3.5) with matrix $L^{(k-1)}$. Recall that $L_{22}^{(k)}$ is a nonsingular diagonal matrix.

4. Estimating the Spectral Condition Number. As it is known, one of the most important parameters in preconditioning technique is the condition number of the considered matrix with respect to the corresponding preconditioner (c.f. [5, 8]). In accordance with the above mentioned, let us determine first the bounds of spectra of the matrices ${B^{(k)}}^{-1}L^{(k)}$. To this end, consider the generalized eigenvalue problem

$$L^{(k)}u = \lambda B^{(k)}u. \tag{4.1}$$

The smallest and the largest eigenvalues of the problem (4.1) are denoted by $\lambda_{min}^{(k)}$ and $\lambda_{max}^{(k)}$ respectively.

Lemma 4.1. For all $k = 1, 2, \ldots, p,$ the largest eigenvalue $\lambda_{max}^{(k)} = 1.$

Proof. Using the technique of transition from the nodal hierarchical basis to the so-called *two-level hierarchical basis* in the space V_k (c.f. [6, 9]), it can be proved that

$$u^T L^{(k)}u \le u^T B^{(k)}u \qquad \forall u \in G_k. \tag{4.2}$$

Note, that a similar inequality has been proved in [6, 10] when constructing preconditioners for finite element matrices. The inequality (4.2) means that for the eigenvalues of the problem (4.1) the estimate $\lambda \le 1$ holds [11].

On the other hand, $\lambda = 1$ is eigenvalue of the problem (4.1). In fact, proceeding from the block structures (3.1) and (3.3) of the matrices $L^{(k)}$ and $B^{(k)}$, respectively, it can be readily seen that $\lambda = 1$ and arbitrary grid function $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, where $u_1 = 0$, $u_2 \neq 0$, are solution of the generalized eigenvalue problem (4.1). Thus, $\lambda_{\max}^{(k)} = 1.$ □

To move further, we make an assumption. Let $d_m^{(p)}$ be the number of points from the set X, which belong to the interval $\tilde{\Delta}_m^{(p)}$ (see (2.3)). We assume that for all values of m the inequalities

$$c_1 \frac{N}{n_p - 1} \le d_m^{(p)} \le c_2 \frac{N}{n_p - 1} \tag{4.3}$$

hold, where c_1 and c_2 are some positive constants.

Lemma 4.2. For all values $k = 1, 2, \ldots, p-3$, is valid the estimate

$$\lambda_{\min}^{(k)} \ge \frac{c_1}{c_2} \cdot \frac{(2 \cdot 2^{p-k} - 1)(2^{p-k} - 4)}{4(2 \cdot 2^{p-k} + 1)(2^{p-k} - 1)}. \tag{4.4}$$

Proof. By Lemma 4.1, $\lambda = 1$ is an eigenvalue of the problem (4.1). Procceding from the block structures (3.1) and (3.3) of the matrices $L^{(k)}$ and $B^{(k)}$, respectively, it can be readily shown that, under the condition $\lambda \neq 1$, the problem (4.1) is reduced to the generalized eigenvalue problem [8]

$$S^{(k)}u_1 = \lambda L^{(k-1)}u_1, \qquad u_1 \in G_k^{(1)}. \tag{4.5}$$

Thus, $\lambda_{\min}^{(k)}$ can be considered as the smallest eigenvalue of the problem (4.5). Hence, this eigenvalue can be computed through the following *generalized Rayleigh quotient*:

$$\lambda_{\min}^{(k)} = \frac{v_1^T S^{(k)} v_1}{v_1^T L^{(k-1)} v_1}, \tag{4.6}$$

where nonzero $v_1 \in G_k^{(1)}$ is the corresponding eigenvector (c.f. [5, 11]). Further, let us define a grid function $v_2 \in G_k^{(2)}$ as follows: $v_2 = -{L_{22}^{(k)}}^{-1} L_{21}^{(k)} v_1$. Then we obtain a grid function $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in G_k$, for which

$$L_{21}^{(k)} v_1 + L_{22}^{(k)} v_2 = 0. \tag{4.7}$$

By construction, $v^T L^{(k)} v = v_1^T S^{(k)} v_1$. Thus,

$$\lambda_{\min}^{(k)} = \frac{v^T L^{(k)} v}{v_1^T L^{(k-1)} v_1}. \tag{4.8}$$

Now consider $(k-1)$-th level of the hierarchical partition of the segment $[a,b]$. We have the intervals $\tilde{\Delta}_m^{(k-1)} \equiv [t_m^{(k-1)}, t_m^{(k-1)} + h_{k-1})$ associated with that level (see (2.3)). As follows from (2.6)−(2.8),

$$v^T L^{(k)} v = \sum_{\tilde{\Delta}_m^{(k-1)}} \sum_{x_i \in \tilde{\Delta}_m^{(k-1)}} \hat{v}^2(x_i), \tag{4.9}$$

where $\hat{v}(x) = prol(v \in G_k : V_k)$. Similarly,

$$v_1^T L^{(k-1)} v_1 = \sum_{\tilde{\Delta}_m^{(k-1)}} \sum_{x_i \in \tilde{\Delta}_m^{(k-1)}} \hat{v}_1^2(x_i), \tag{4.10}$$

where $\hat{v}_1(x) = prol(v \in G_k^{(1)} : V_{k-1})$. As a result, from (4.8)−(4.10) we get that

$$\lambda_{\min}^{(k)} = \frac{\sum\limits_{\tilde{\Delta}_m^{(k-1)}} \sum\limits_{x_i \in \tilde{\Delta}_m^{(k-1)}} \hat{v}^2(x_i)}{\sum\limits_{\tilde{\Delta}_m^{(k-1)}} \sum\limits_{x_i \in \tilde{\Delta}_m^{(k-1)}} \hat{v}_1^2(x_i)}.$$

Taking into account the above assumption that each interval $\Delta_m^{(p)}$ contains at least one point from the set X, it can be proved that

$$\lambda_{\min}^{(k)} \geq \frac{\sum\limits_{x_i \in \tilde{\Delta}_m^{(k-1)}} \hat{v}^2(x_i)}{\sum\limits_{x_i \in \tilde{\Delta}_m^{(k-1)}} \hat{v}_1^2(x_i)}, \tag{4.11}$$

where $\tilde{\Delta}_m^{(k-1)}$ is an interval, in which function $\hat{v}_1(x)$ is not identically zero. Let us introduce the notations

$$J_1^{(k)} \equiv \sum_{x_i \in \tilde{\Delta}_m^{(k-1)}} \hat{v}^2(x_i), \quad J_2^{(k)} \equiv \sum_{x_i \in \tilde{\Delta}_m^{(k-1)}} \hat{v}_1^2(x_i). \tag{4.12}$$

Then the inequality (4.11) can be written as follows:

$$\lambda_{\min}^{(k)} \geq \frac{J_1^{(k)}}{J_2^{(k)}}. \tag{4.13}$$

So, our task is to estimate the quantities $J_1^{(k)}$ and $J_2^{(k)}$. Let start with $J_1^{(k)}$. First we perform some auxiliary constructions and introduce corresponding notations. We have the nodes $t_m^{(k-1)} \in \sigma_k^{(1)}$, $t_{m'}^{(k)} \equiv t_m^{(k-1)} + h_k \in \sigma_k^{(2)}$, $t_{m''}^{(k-1)} \equiv t_m^{(k-1)} + h_{k-1} \in \sigma_k^{(1)}$

of the grid σ_k associated with the considered interval $\Delta_m^{(k-1)} = (t_m^{(k-1)}, t_m^{(k-1)} + h_{k-1})$. Correspondingly, let $\varphi_1(x) \equiv \varphi_m^{(k)}(x)$, $\quad \varphi_2(x) \equiv \varphi_{m''}^{(k)}(x)$, $\quad \varphi_3(x) \equiv \varphi_{m'}^{(k)}(x)$.

According to the procedure of grid generation, the interval $\Delta_m^{(k-1)}$ contains 2^{p-k+1} intervals of the p-th level of the hierarchical partition of the segment $[a,b]$. The points $\mu_i = t_m^{(k-1)} + ih_p$, $\quad i = 0, 1, \ldots, 2^{p-k+1}$, are nodes of the p-th level. The values of piecewise linear functions can be easily computed:

$$\begin{gathered}\varphi_1(\mu_i) = 1 - 2^{-(p-k)}i, \quad \varphi_3(\mu_i) = 2^{-(p-k)}i, \quad i = 0, 1, \ldots, 2^{p-k}, \\ \varphi_2(\mu_i) = -1 + 2^{-(p-k)}i, \quad \varphi_3(\mu_i) = 2 - 2^{-(p-k)}i, \\ i = 2^{p-k}, 2^{p-k}+1, \ldots, 2^{p-k+1}.\end{gathered} \tag{4.14}$$

In each of the intervals $[\mu_i, \mu_{i+1})$, $i = 1, 1, \ldots, 2^{p-k+1} - 1$, we choose exactly s points of the set X. In accordance with (4.3) and (2.2),

$$c_1 \frac{N}{2^p(n_0-1)} \leq s \leq c_2 \frac{N}{2^p(n_0-1)}. \tag{4.15}$$

Let us denote the set of selected points by $X^{(k)}$. Obviously,

$$J_1^{(k)} \geq \sum_{x_i \in X^{(k)}} \hat{v}^2(x_i). \tag{4.16}$$

Now we introduce the following notations:
$w_1 \equiv \hat{v}(t_m^{(k-1)}) = \hat{v}_1(t_m^{(k-1)})$, $w_2 \equiv \hat{v}(t_{m''}^{(k-1)}) = \hat{v}_1(t_{m''}^{(k-1)})$, $w_3 \equiv \hat{v}(t_{m'}^{(k)})$.
Then we can write the piecewise linear function $\hat{v}(x)$ in the segment $[t_m^{(k-1)}, t_{m''}^{(k-1)}]$ in the following form: $\hat{v}(x) = w_1\varphi_1(x) + w_2\varphi_2(x) + w_3\varphi_3(x)$. Proceeding from (4.16), we get

$$J_1^{(k)} \geq s_{11}w_1^2 + s_{22}w_2^2 + s_{33}w_3^2 + 2s_{12}w_1w_2 + 2s_{13}w_1w_3 + 2s_{23}w_2w_3, \tag{4.17}$$

where

$$s_{\alpha\beta} \equiv \sum_{x_i \in X^{(k)}} \varphi_\alpha(x_i)\varphi_\beta(x_i), \quad \alpha, \beta = 1, 2, 3. \tag{4.18}$$

It can be easily seen that $s_{12} = 0$. Above we have established the relation (4.7) for our grid function $v \in G_k$. Then, using the formula (2.8) and the notation (4.18), we obtain the relation

$$s_{13}w_1 + s_{23}w_2 + s_{33}w_3 = 0. \tag{4.19}$$

So, from (4.17) and (4.19) we arrive at the inequality $J_1^{(k)} \geq s_{11}w_1^2 + s_{22}w_2^2 - s_{33}w_3^2$. Again, using the relation (4.19), let us eliminate the quantity w_3 in the right-hand side of the last inequality. As a result we get

$$J_1^{(k)} \geq \left(s_{11} - 2\frac{s_{13}^2}{s_{33}}\right)w_1^2 + \left(s_{22} - 2\frac{s_{23}^2}{s_{33}}\right)w_2^2. \tag{4.20}$$

Thus, it remains to estimate the quantities $s_{\alpha\beta}$ defined in (4.18). For example, we estimate s_{11}. Using the expressions (4.14), by simple calculation we obtain

$$s_{11} = \sum_{x_i \in X^{(k)}} \varphi_1^2(x_i) \geq s \sum_{i=1}^{2^{p-k}-1} \varphi_1^2(\mu_i) = s\,\frac{(2^{p-k}-1)(2 \cdot 2^{p-k}-1)}{6 \cdot 2^{p-k}}. \tag{4.21}$$

The remaining quantities are estimated similarly, namely,

$$s_{22} \geq s\,\frac{(2^{p-k}-1)(2\cdot 2^{p-k}-1)}{6\cdot 2^{p-k}}, \quad s_{33} \geq s\,\frac{(2^{p-k}-1)(2\cdot 2^{p-k}-1)}{3\cdot 2^{p-k}} \tag{4.22}$$

and

$$s_{\alpha 3} \leq s\,\frac{(2^{p-k}+2)(2\cdot 2^{p-k}-1)}{12\cdot 2^{p-k}}, \quad \alpha = 1,2. \tag{4.23}$$

From (4.20)−(4.23) we obtain the inequality

$$J_1^{(k)} \geq s\,\frac{(2\cdot 2^{p-k}-1)(2^{p-k}-4)}{8(2^{p-k}-1)}(w_1^2+w_2^2).$$

Taking into account (4.15),we obtain

$$J_1^{(k)} \geq c_1\frac{N}{2^p(n_0-1)}\,\frac{(2\cdot 2^{p-k}-1)(2^{p-k}-4)}{8(2^{p-k}-1)}(w_1^2+w_2^2). \tag{4.24}$$

By a similar reasoning, we obtain the inequality

$$J_2^{(k)} \leq c_2\frac{N}{2^p(n_0-1)}\,\frac{2\cdot 2^{p-k}+1}{2}(w_1^2+w_2^2). \tag{4.25}$$

Finally, the required estimate (4.4) follows from the inequalities (4.13), (4.24) and (4.25). □

It is easy to verify that the quantity in the right-hand side of the inequality (4.4) increases with k decreasing.

Now we can estimate the *spectral condition number* of the matrices $B^{(k)^{-1}}L^{(k)}$, which is defined as follows: $\kappa(B^{(k)^{-1}}L^{(k)}) \equiv \lambda_{\max}^{(k)}/\lambda_{\min}^{(k)}$ ([8, 11]).

As a direct consequence of Lemmas 4.1 and 4.2, we get the following statement.

Theorem. For all values $k = 1,2,\ldots,p-3$, the estimate

$$\kappa(B^{(k)^{-1}}L^{(k)}) \leq \frac{c_2}{c_1}\,\frac{4(2\cdot 2^{p-k}+1)(2^{p-k}-1)}{(2\cdot 2^{p-k}-1)(2^{p-k}-4)} \tag{4.26}$$

is valid.

Consider closely the quantity $\delta_k \equiv \dfrac{4(2\cdot 2^{p-k}+1)(2^{p-k}-1)}{(2\cdot 2^{p-k}-1)(2^{p-k}-4)}$ from the right-hand side of the inequality (4.26). The strightforward calculation shows that

$$4 < \delta_1 < \delta_2 < \cdots < \delta_{p-3} = \frac{119}{15} \approx 7.94\,. \tag{4.27}$$

Thus, the spectral condition number of the matrices $B^{(k)^{-1}}L^{(k)}$ decreases with k decreasing. Moreover, as follows from (4.26) and (4.27), the condition numbers are bounded from above by a value independent of the number of refinement levels of the initial grid.

5. Concluding Remarks. In this paper we have discussed an idea of constructing algebraic two-level preconditioners for *NS*-matrices arising in data fitting by least squares method using piecewise linear basis functions. The results obtained can be useful for constructing multilevel preconditioners. They also serve as a basis when considering more smooth approximating functions. We will study these issues in subsequent publications.

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