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Mathematics

OPERATOR ANALOGUE OF BERNSTEIN THEOREM

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In this article obtained operator analogue of well-known S. Bernstein Theorem about approximation on the real axis of a bounded and uniformly continuous function by entire functions of Bernstein space.

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Let $\mathcal{A}$ is involutive and commutative Banach algebra with unit. We denote by $M_{\mathcal{A}}$ a maximal ideal space of $\mathcal{A}$ and by $Sym(\mathcal{A}^*)$ the set of continuous and self-adjoint functional space. We assume that $\mathcal{A}$ disjoint elements of $\mathcal{A}$ (i.e. involution is continuous and symmetric in sense that $\varphi(x^*) = \widehat{\varphi(x)}$). It is well-known, if $\mathcal{K}$ is a set of all positive functional spaces on $\mathcal{A}$ with $\Phi(1) \leq 1$, and $\mathcal{M}$ is a set of all positive regular Borel measures μ on $M_{\mathcal{A}}$ with $\mu(M_{\mathcal{A}})$, then the formula of Bochner $\Phi(x) = \int_{M_{\mathcal{A}}} \widehat{x} d\mu$ establishes one-to-one correspondence between $\mathcal{K}$ and $\mathcal{M}$, which carries extreme points to extreme points [1]. Consequently, the multiplicative linear functional on $\mathcal{A}$ are precisely the extreme points of $\mathcal{K}$ set. Recall that commutative set is called normal, if from condition $S \subset \mathcal{A}$ follows $x^* \in S$. Further assumed that commutative algebra $\mathcal{A}$ is normal subalgebra of the algebra of all bounded linear operators $BL(H)$ acting in Hilbert space H.

Let $\Omega_{\mathcal{A}} \subset \mathcal{A}$ is domain, and $f : \Omega_{\mathcal{A}} \to \mathcal{A}$ is holomorphic by Lorch mapping [2,3]. If $\Omega_{\mathcal{A}} = \mathcal{A}$, then mapping f called an entire by Lorch mapping.

Let $f : \mathcal{A} \to \mathcal{A}$ is a entire by Lorch mapping, than for every $\varphi \in M_{\mathcal{A}}$ exist a unique entire function $g_\varphi : \mathbb{C} \to \mathbb{C}$, which diagram $\varphi \circ f = g_\varphi \circ \varphi$ is commutative [4] entire by Lorch mapping f with a form

$$f(\omega) = \sum_{i=0}^{\infty} \frac{c_k w^k}{k!},$$

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where $\overline{\lim\limits_{k\to\infty}}\sqrt[k]{\|c_k\|} = \widehat{\sigma} < \infty$ will be called a mapping of exponent type not exceeding $\widehat{\sigma}$. Notice, if $c_k = c^k$, where c is any fixed element of algebra $\mathcal{A}$, then $\widehat{\sigma} = \rho(c)$, where $\rho(c)$ is spectral radius of an element c.

Denote by $E_{\widehat{\sigma}}(\mathcal{A})$ the space of all entire by Lorch mapping of an exponential type not exceeding $\widehat{\sigma}$.

Denote by $\mathcal{H}(\mathcal{A})$ the space of self-adjoint (in this case it is the same as Hermite) operators in algebra. Let $C_b(\mathcal{H}(\mathcal{A}),\mathcal{A})$ is a set of all bounded functions, and $C_{b,u}(\mathcal{H}(\mathcal{A}),\mathcal{A})$ is a set of all bounded uniformly continuous functions $\mathcal{A}$-valued mappings from $\mathcal{H}(\mathcal{A})$ to $\mathcal{A}$. It is easy to see that sup-norm

$$\|g\|_\infty = \sup_{h\in\mathcal{H}(\mathcal{A})} \|g\|_{\mathcal{A}} < \infty$$

make $C_b(\mathcal{H}(\mathcal{A}),\mathcal{A})$ in a Banach algebra, and $C_{b,u}(\mathcal{H}(\mathcal{A}),\mathcal{A})$ in his subalgebra. The space of Bernstein $B_{\widehat{\sigma}}(\mathcal{A})$ considered as a set of entire by Lorch mapping $f \in E_{\widehat{\sigma}}(\mathcal{A})$, satisfying condition

$$\|f\|_\infty = \sup_{h\in\mathcal{H}(\mathcal{A})} \|f(h)\|_{\mathcal{A}},$$

is a Banach space with norm $\|.\|_\infty$.

Let $f \in B_{\widehat{\sigma}}(\mathcal{A})$ and $\varphi \in M_{\mathcal{A}}$, then

$$f_\varphi(z) = \varphi(f(\omega)) = f(\varphi(\omega)) = f(\varphi(\omega)) \in B_{\widehat{\sigma}}(\mathbb{C}),$$

because of

$$|f_\varphi(x)| = |\varphi(f(h))| \leq \|f(h)\| < \infty,$$

we have

$$|f_\varphi(z)| = |\varphi(f(\omega))| = \sum_{i=0}^{\infty} \frac{\varphi(c_k)z^k}{k!},$$

when

$$\overline{\lim_{k\to\infty}}\sqrt[k]{|\varphi(c_k)|} = \overline{\lim_{k\to\infty}}\sqrt[k]{|\widehat{c_k}(\varphi)|} \leq \overline{\lim_{k\to\infty}}\sqrt[k]{\|\widehat{c_k}\|_\infty} = \overline{\lim_{k\to\infty}}\sqrt[k]{\|\widehat{c_k}\|} = \widehat{\sigma}.$$

Applying classical Bernstein inequality [5, 6]

$$|f'_\varphi(x)| \leq \widehat{\sigma}\sup_{\mathbb{R}}|f_\varphi(x)|,$$

and by a definition of a derivative by Lorch the next inequality holds:

$$|\varphi'(f(h))| \leq \widehat{\sigma} \sup_{h\in\mathcal{H}(\mathcal{A})} |\varphi(f(h))| = \widehat{\sigma} \sup_{h\in\mathcal{H}(\mathcal{A})} |\widehat{f(h)}(\varphi)| \leq \widehat{\sigma} \sup_{h\in\mathcal{H}(\mathcal{A})} \|f(h)\|.$$

Using Banach–Steinhaus Theorem [1], we get

$$\|f'(h)\| \leq \alpha\widehat{\sigma} \sup_{h\in\mathcal{H}(\mathcal{A})} \|f(h)\|,$$

where α is a positive number depend on $M_{\mathcal{A}}$.

The following theorem is true:

Theorem. If $F \in C_{b,u}(\mathcal{H}(\mathcal{A}),\mathcal{A})$, then exist a sequence of entire by Lorch mappings $f_k \in B_{\widehat{\sigma_k}}(\mathcal{A})$ such that $\|F - f_k\|_\infty \longrightarrow 0$ when $k \to \infty$.

Proof. From classic Bernstein Theorem follows, that for every fixed $F \in C_{b,u}(\mathcal{H}(\mathcal{A}),\mathcal{A})$ and for every fixed $\varphi \in M_{\mathcal{A}}$, $F_\varphi \in C_{b,u,\varphi}(\mathbb{R},\mathbb{C})$ exists entire functions $f_{\varphi,k}$ from spaces $B_{\sigma_{k,\varphi}}$ correspondingly, such that

$$\sup_{\mathbb{R}} |F_\varphi(x) - f_{\varphi,k}(x)| \to 0 \text{ when } k \to \infty.$$

On the other hand, it means that exists a mappings $f_k \in B_{\widehat{\sigma_k}}(\mathcal{A})$, where $f_{\varphi,k} = \varphi(f_k)$. By the Banach–Steinhaus Theorem,

$$\sup_{h\in\mathcal{H}(\mathcal{A})} |\varphi((F(h) - f_k(h))| \longrightarrow 0 \text{ when } k \to \infty.$$

From which follows that

$$\|\widehat{F-f}\|_\infty \to 0 \text{ when } k \to \infty. \quad (1)$$

But $\mathcal{A}$ is a normal subalgebra in $BL(H)$ and Gelfand's transform is an isometric isomorphism. This means that

$$\|\widehat{F} - \widehat{f_k}\| = \|F - f_k\|$$

for every k. Then from (1) follows

$$\|F - f_k\|_\infty \to 0 \text{ when } k \to \infty. \qquad \square$$

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