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M a t h e m a t i c s

ON A RECURSIVE APPROACH TO THE SOLUTION OF MINLA PROBLEM

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In this paper a recursive approach is suggested for the problem of Minimum Linear Arrangement (MINLA) of a graph by length. A minimality criterion of an arrangement is presented, from which a simple proof is obtained for the polynomial solvability of the problem in the class of bipartite, Γ-oriented graphs.

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1. Introduction. We will assume that the graphs considered in this paper are finite, oriented and do not contain multiple edges or loops. For a graph G, let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. For a vertex $v \in V(G)$, let Γ_v^- and Γ_v^+ denote the sets of ancestors and predecessors of v respectively: $\Gamma_v^- = \{u \in V/(u,v) \in E(G)\}$, $\Gamma_v^+ = \{u \in V/(v,u) \in E(G)\}$. An oriented graph $G(V,E)$ is called Γ-oriented, if for any vertices $u,v \in V(G)$ either $\Gamma_v^+ \subseteq \Gamma_u^+$ or $\Gamma_u^+ \subseteq \Gamma_v^+$. The terms and concepts, which are not defined here can be found in [1]. The problem of a minimum linear arrangement (MINLA) of oriented graphs is defined as follows:

P r o b l e m. For a given oriented graph $G(V,E)$ construct a one to one function $f : V \mapsto \{1,,|V(G)|\}$ such that the following two conditions are satisfied:

$$\forall (u,v) \in E(G),\ f(u) < f(v),$$
$$\sum_{(u,v)\in E(G)} (f(v) - f(u)) \to \min. \tag{1.1}$$

Any function, satisfiing (1.1), the acceptability condition, is called a labeling function for the graph G. We denote by $F(G)$ the set of all labeling functions of the graph G. The length $L(G,f)$ of the arrangement $f \in F(G)$ is defined as follows:

$$L(G,f) = \sum_{(u,v)\in E(G)} (f(v) - f(u)). \tag{1.2}$$

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Let $L(G) = \min_{f \in F(G)} L(G, f)$. Define $M(G) = \{f \in F(G)/L(G) = L(G, f)\}$.

It is clear that a vertex $v \in V(G)$ has different impact on the length of the arrangement depending on $f \in F(G)$, so let us introduce a weight function $W : V(G) \times F(G) \mapsto Z_+$ by

$$W(v, f) = L(G, f) - L(G \backslash v, f_v), \tag{1.3}$$

where $G\backslash v$ is a graph obtained from G by removing the vertex v and f_v is the arrangement for $G\backslash v$ defined by:

$$f_v(u) = \begin{cases} f(u), & \text{if} \quad f(u) < f(v), \\ f(u) - 1, & \text{if} \quad f(v) < f(u). \end{cases} \tag{1.4}$$

Obviously $f_v \in F(G\backslash v)$, since the acceptability condition is inherited from f. Let us define the minimum impact of the vertex $v \in V(G)$ as follows:

$$W_*(v) = \min_{f \in F(G)} W(v, f).$$

2. The Main Result. It is known that MINLA problem for oriented graphs is NP complete [2], and it remains NP complete for transitive oriented, bipartite graphs [3]. It is also known [4] that for any bipartite, Γ-oriented graph $G(V,E)$ with $V = X \cup \cup Y$, $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_m\}$, where $|\Gamma^+_{x_1}| \geq |\Gamma^+_{x_2}| \geq ... \geq |\Gamma^+_{x_n}|$, $|\Gamma^-_{y_1}| \leq |\Gamma^-_{y_2}| \leq ... \leq |\Gamma^-_{y_m}|$, there exists a minimum linear arrangement f of the following kind:

$$x_n \; x_{n-1} \; ... \; x_1 \; y_m \; y_{m-1} \; ... \; y_1. \tag{2.1}$$

Below we present a new approach to the solved problem of MINLA of bipartite, Γ-oriented graphs (see [4]). The basic idea of the new approach is formulated in Lemma 1. Suppose that a labeling function f of some graph G satisfies the following conditions:

$$\exists v \in V(G) \text{ with } W(v, f) = W_*(v), \tag{2.2}$$

$$f_v \in M(G\backslash v). \tag{2.3}$$

L e m m a 1 . Any arrangement $f \in F(G)$ satisfying the conditions (2.2) and (2.3) is a minimum arrangement for G.

P r o o f . From (1.3), $L(G, f) = W(v, f) + L(G\backslash v, f_v) = W_*(v) + L(G\backslash v, f_v) = = W_*(v) + L(G\backslash v)$. Since for any arrangement $h \in F(G)$ the inequalities $W(v,h) \geq \geq W_*(v)$ and $L(G\backslash v, h) \geq L(G\backslash v)$ hold by definition, we can conclude that

$$L(G, h) = W(v, h) + L(G\backslash v, h_v) \geq W_*(v) + L(G\backslash v) = L(G, f).$$

So, for any $h \in F(G)$, $L(G, h) \geq L(G, f)$, which means that $f \in M(G)$. □

Remark 1. Lemma 1 can be applied for arrangements of non-oriented graphs.

L e m m a 2 . For any bipartite, Γ-oriented graph G and any arrangement $h \in F(G)$, $W(x_n, h) \geq \dfrac{|\Gamma^+_{x_n}|(|\Gamma^+_{x_n}| + 1)}{2} + (n-1)|\Gamma^+_{x_n}|$, and $W(x_n, f) = W_*(x_n)$, where f is given by (2.1).

P r o o f . Since $h \in F(G)$ we have that there are vertices $x_{i_1}, x_{i_2}, ..., x_{i_k}$ such that $h(x_{i_j}) < h(x_n)$, $j = 1, ..., k$, and vertices $x_{i_{k+1}}, ..., x_{i_{n-1}}$ such that $h(x_{i_j}) > h(x_n)$, $j = k+1, ..., n-1$, and for all vertices $y \in \Gamma^+_{x_n}$, $h(y) > h(x_i), i = 1, ..., n$.

Now let estimate the impact of x_n on the length of the arrangement. By removing x_n, we also remove $|\Gamma^+_{x_n}|$ edges, the i -th of them, $1 \le i \le |\Gamma^+_{x_n}|$, has the length no less than $n-1-k+i$. Moreover, we are shortening by 1 all edges from $x_{i_1}, x_{i_2},, x_{i_k}$ to $\Gamma^+_{x_n}$. So, we are shortening at least $k|\Gamma^+_{x_n}|$ edges by 1. We have

$$L(G\backslash x_n, h_{x_n}) \le L(G,h) - \frac{|\Gamma^+_{x_n}|(|\Gamma^+_{x_n}|+1)}{2} - (n-1)|\Gamma^+_{x_n}|. \tag{2.4}$$

Using $W(x_n,h) = L(G,h) - L(G\backslash x_n, h_{x_n})$ and (2.4), we obtain

$$W(x_n,h) \ge L(G,h) - (L(G,h) - \frac{|\Gamma^+_{x_n}|(|\Gamma^+_{x_n}|+1)}{2} - (n-1)|\Gamma^+_{x_n}|) \text{ and}$$

$$W(x_n,h) \ge \frac{|\Gamma^+_{x_n}|(|\Gamma^+_{x_n}|+1)}{2} + (n-1)|\Gamma^+_{x_n}|. \tag{2.5}$$

Since (2.5) holds for any arrangement $h \in F(G)$, then

$$W_*(x_n) \ge \frac{|\Gamma^+_{x_n}|(|\Gamma^+_{x_n}|+1)}{2} + (n-1)|\Gamma^+_{x_n}|.$$

On the other hand, it is easy to see that

$$W(x_n,f) = \frac{|\Gamma^+_{x_n}|(|\Gamma^+_{x_n}|+1)}{2} + +(n-1)|\Gamma^+_{x_n}|.$$

Consequently $W(x_n,f) = W_*(x_n)$. □

T h e o r e m . For an arbitrary bipartite, Γ-oriented graph G, f (the labeling function f given in (2.1)) is a minimum linear arrangement.

P r o o f . We prove the theorem by induction on the number of vertices of X. It is easy to see that for $|X| = 1$, f is a minimum linear arrangement. Let us assume that the theorem holds for $|X| = n-1$ and let us prove it for $|X| = n$. From Lemma 2 we have that $W(x_n,f) = W_*(x_n)$. Since in f_{x_n} the decreasing order of degrees of the vertices $y_m, y_{m-1}, \ldots, y_1$ in $G\backslash x_n$ is kept, then, by the assumption of the induction, we have $f_{x_n} \in M(G\backslash x_n)$. It means that f satisfies the conditions of Lemma 1 and, thus, f is a minimum linear arrangement of the graph G. □

Remark 2. Similarly, it can be shown that f', given by

$$x_1\ y_1 \ldots y_{t_1}\ x_2\ y_{t_1+1} \ldots x_{n-1}\ y_{t_{n-2}+1} \ldots y_{t_{n-1}}\ x_n\ y_{t_{n-1}+1} \ldots y_m,$$

where $t_i = |\Gamma^+_{x_1}| - |\Gamma^+_{x_{i+1}}|$, also satisfies the conditions of Lemma 1 and $f' \in M(G)$.

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