PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2014, № 3, p. 24–30

M a t h e m a t i c s

THE C^*-ALGEBRA $\mathfrak{T}_m$ AS A CROSSED PRODUCT

K. H. HOVSEPYAN *

Kazan State Power Engineering University, Russian Federation

In this paper we consider the C^*-subalgebra $\mathfrak{T}_m$ of the Toeplitz algebra $\mathfrak{T}$ generated by monomials, which have an index divisible by m. We present the algebra $\mathfrak{T}_m$ as a crossed product: $\mathfrak{T}_m = \varphi(\mathcal{A}) \times_{\delta_m} \mathbb{Z}$, where $\mathcal{A} = C_0(\mathbb{Z}_+) \oplus \mathbb{C}I$ is C^*-algebra of all continuous functions on $\mathbb{Z}_+$, which have a finite limit at infinity. In the case $m = 1$ we obtain that $\mathfrak{T} = \varphi(\mathcal{A}) \times_{\delta_1} \mathbb{Z}$, which is an analogue of Coburn's theorem.

MSC2010: 22D05.

***Keywords*:** index of monomial, coefficient algebra, crossed product, finitely representable, Toeplitz algebra, C^*-algebra, transfer operator.

1. Introduction. The crossed product of a C^*-algebra A by an automorphism $\alpha\colon A \to A$ is defined as a universal C^*-algebra generated by a copy of A and a unitary element U, satisfying the relations

$$\alpha(a) = UaU^*, \alpha^{-1}(a) = U^*aU,\ a \in A.$$

In order to define the crossed product of a C^*-algebra A by an endomorphism, that is, to replace an automorphism with an endomorphism in the above definition, it is necessary answer the following questions:

(1) What criterias should the element U satisfy?

(2) What should be used in place of α?

There is a number of methods to construct a new C^*-algebra (which is an extension of A) using a given C^*-algebra A and an endomorphism $\alpha\colon A \to A$. As a result of an extension we get a new algebra, which is called crossed product, and this new algebra contains the algebra A as a subalgebra. In the works of many authors [1–6] there are diverse methods of constructing crossed product, where they find answers to the questions above from different aspects.

In the paper [4] Exel investigates the notion of the transfer operator L of C^* dynamical system (A, α), which satisfies certain conditions, and, as a matter of fact

* E-mail: karen.hovsep@gmail.com

plays role of α^{-1}. In this case crossed product is defined as a universal C^*-algebra generated by a copy of A and an unitary element S, which generates the transfer operator L, and satisfies certain conditions.

In the work [5] Lebedev and Odzijevich give an extension of C^*-algebra by partial isometries. They introduce the notion of so-called coefficient algebra, which plays a crucial role both in the extensions of C^*-algebras by partial isometries and in the construction of the crossed product by an endomorphism. As we will see, initially given algebra will be a coefficient algebra of crossed product.

A new method of an extension of C^*-algebra by ismotries was introduced by Pashque [2]. He gives conditions, in which the C^*-algebra $C^*(A,S)$, which is the extension of C^*-algebra by isomorphism S, both is isomorphic to $A \times_\alpha S$, and as a C^*-algebra is simple.

In the paper [6] a new method of an extension and a construction of a crossed product is given. A crossed product is constructed by coefficient an algebra, using the transfer operator introduced by Exel. In the paper authors prove, that the crossed products constructed in different ways are isomorphic to the crossed product introduced in [6].

In this paper we begin the study of C^*-subalgebra of the Toeplitz algebra $\mathcal{T}$, generated by monomials, which have an index divisible by m. We denote that C^*-algebra by $\mathcal{T}_m$. The aim of this work is to show that: for any $m \in \mathbb{N}$ the C^*-algebra $\mathcal{T}_m$ is represented as a crossed product of the C^*-algebra $C_0(\mathbb{Z}_+) \oplus \mathbb{C}I$ of all continuous functions on $\mathbb{Z}_+$, which have a finite limit at infinity by the endomorphism $\delta_m(a) = T^m a T^{*m}$. The construction of crossed product, introduced in [6] is used.

2. Preliminaries and Definition of the Crossed Product. In this part we recall some definitions and facts concerning transfer operators and coefficient algebras and, as we will see, it will be used in the following parts of this work. Presented definitions and results are taken from [4–6].

Let B be a C^*-algebra with an identity 1 and let $\delta_*: B \to B$ be an endomorphism of this algebra. A linear map $\delta_*: B \to B$ is called a *transfer operator* for the pair (B,δ), if it is continuous and positive and such that

$$\delta_*(\delta(a)b) = a\ \delta(b),\ \forall a,b\ \in B.$$

Transfer operator is called *full*, if

$$\delta\delta_*(a) = \ \delta(1)a\ \delta(1),\ a \in B.$$

Let $A \subset B(H)$ be a *-subalgebra containing the identity $I \in B(H)$, and let $V \in B(H)$. We call A the coefficient algebra of the C^*-algebra $C^*(A,V)$ generated by A and V, if A and V satisfy the following three conditions:

$$Va = VaV^*V,\ a \in A; \tag{2.1}$$

$$VaV^* \in A,\ a \in A; \tag{2.2}$$

$$V^*aV \in A,\ a \in A. \tag{2.3}$$

Definition 2.1. Let δ be an endomorphism of an (abstract) unital C^*-algebra A. We say that the pair (A,δ) is finitely representable, if there exists a triple (H,π,U) consisting of a Hilbert space H, a faithful representation $\pi\colon A\to B(H)$ and a linear continuous operator $U\colon H\to H$ such that for every $a\in A$ the following conditions are satisfied:

$$\pi(\delta(a)) = U\pi(a)U^*,\ U^*\pi(a)U\in\pi(A), \tag{2.4}$$

$$U\pi(a) = \pi(\delta(a))U,\ a\in A. \tag{2.5}$$

That is, $\pi(A)$ is the coefficient algebra for $C^*(\pi(A),U)$ under the fixed endomorphism $U\cdot U^*$.

In this case we also say that A is a coefficient algebra associated with δ.

Theorem 2.1 ([6], Theorem 3.1). A pair (A,δ) is finitely representable, if and only if there exists a full transfer operator δ_*.

Definition 2.2 (Crossed product). Let (A,δ) be a finitely representable pair. The crossed product of A and δ (which we denote by $A\times_\delta\mathbb{Z}$) is the universal unital C^*-algebra generated by a copy of A and a partial isometry U satisfies to the relations

$$\delta(a) = UaU^*,\ \delta_*(a) = U^*aU,\ a\in A,$$

where δ_* is the full transfer operator for (A,δ). The algebra A will be called the coefficient algebra for $A\times_\delta\mathbb{Z}$.

Lemma 2.1 ([6], Proposition 2.3). Let A be the coefficient algebra of $C^*(A,V)$. Then the vector space B_0 consisting of finite sums

$$x = V^{*N}a_{-N}+\ldots+V^*a_{-1}+a_0+a_1V+\ldots+a_NV^N, \tag{2.6}$$

where $a_k\in A$, $N\in\mathbb{N}\cup 0$, is a dense *-subalgebra of $C^*(A,V)$.

Definition 2.3. We say that a C^*-algebra $C^*(A,V)$, mentioned in Definition 2.1, possesses property (*), if for any $x\in B_0$ (given by (2.6)), the inequality

$$\parallel a_0 \parallel\leq\parallel x \parallel \tag{2.7}$$

holds.

3. The C^*-Algebra $\mathcal{T}_m$. Let $l^2(\mathbb{Z}_+)$ be a Hilbert space of all complex-valued functions on $\mathbb{Z}_+$ satisfying

$$f\colon \mathbb{Z}_+\to\mathbb{C},\sum_{n=0}^{\infty}|f(n)|^2<\infty.$$

The set of the functions $\{e_n\}_{n=0}^{\infty}$, $e_n(m)=\delta_{n,m}$, forms an orthonormal basis in $l^2(\mathbb{Z}_+)$, where $\delta_{n,m}$ is the Kronecker symbol.

Let T be the right shift operator on $l^2(\mathbb{Z}_+)$, that is

$$Te_k = e_{k+1}.$$

Obviously, $T^*T=I$, where T^* is the adjoint of T, that is the left shift operator.

Denote by $B(l^2(\mathbb{Z}_+))$ the algebra of all bounded linear operators on $l^2(\mathbb{Z}_+)$.

In this way TT^* is a projection on $l_2(Z_+\{0\})$. Therefore, the semigroup generated by T and T^* forms bicyclic semigroup. Hence, each finite product of the operators T and T^* has a form T^nT^{*m}, where $n,m \in Z_+$. Such elements are called *monomials*, and the number $n-m$ is an index of the monomial T^nT^{*m} and is denoted $\mathrm{ind}(TT^{*m})$. Thus, the Toeplitz algebra, which is denoted by $\mathcal{T}$, is a uniform closure of the involutive subalgebra generated by finite linear combinations of monomials of $B(l^2(\mathbb{Z}_+))$. Denote by $\mathcal{T}_m$ C^*-subalgebra of the Toeplitz algebra, generated by monomials, which have an index divisible by m.

4. The C^*-Algebra $\mathcal{T}_m$ as a Crossed Product. Let us consider a C^*-algebra $\mathcal{A} = C_0(\mathbb{Z}_+) \oplus \mathbb{C}I$, i. e. the algebra of continuous functions on $\mathbb{Z}_+$, which have finite limits at infinity.

Denote by the $\mathcal{P}$ commutative C^*-subalgebra of $\mathcal{T}_m$, which consists of the projections

$$\mathcal{P} = [I, TT^*, ..., T^mT^{*m}, ...].$$

We denote the projections by $P_i = T^iT^{*i}$, where $i = 0, 1, ...$

L e m m a 4 . 1 . There exists an isomorphism between C^*-algebras $\mathcal{P}$ and $\mathcal{A}$:

$$\mathcal{P} \cong \mathcal{A}.$$

P r o o f . Define a homomorphism $\varphi \colon \mathcal{A} \to \mathcal{P}$ in the following way:

$$\varphi(f) = \sum_{n=0}^{\infty} f(n)Q_n,$$

where $Q_i = P_{i-1} - P_i$, $i = 1, 2, ...$, are orthonormal projectors.

It is easy to see that $\varphi(fg) = \sum\limits_{n=0}^{\infty}(fg)(n)Q_n = \sum\limits_{n=0}^{\infty} f(n)g(n)Q_n = \varphi(f)\varphi(g)$, for any $f, g \in \mathcal{A}$. Clearly, $\varphi(I) = I$, that is φ is an isomorphism. □

Consider a C^*-dynamical system $(\mathcal{P}, \delta_m)$, where the endomorphism $\delta_m \colon \mathcal{P} \to \mathcal{P}$ is defined as follows:

$$\delta_m(a) = T^maT^{*m},\ a \in \mathcal{P}. \tag{4.1}$$

Since T^m is an isomorphic operator obviously the pair $(\mathcal{P}, \delta_m)$ satisfies the conditions (2.2) – (2.4). Denote

$$\delta_{*m} := T^{*m}aT^m,\ a \in \mathcal{P}. \tag{4.2}$$

L e m m a 4 . 2 . The map δ_{*m} defined above is a full transfer operator for the pair $(\mathcal{P}, \delta_m)$.

P r o o f . It is evident that δ_{*m} is a positive operator. Moreover,

$$\delta_m(\delta_{*m}(a)b) = T^{*m}(T^maT^{*m}b)T^m = aT^{*m}bT^m = a, \quad a, b \in \mathcal{P}.$$

This means that δ_{*m} is a transfer operator fot the pair $(\mathcal{P}, \delta_m)$. Now we show that δ_{*m} is a full transfer operator:

$$\delta_m(\delta_{*m}(a)) = T^mT^{*m}aT^mT^{*m} = \delta_m(1)a\delta_m(1). \qquad \square$$

It follows from the Lemma 4.2 and Theorem 2.1, that a pair $(\mathcal{P}, \delta_m)$ is finitely representable. Hence we have that the pair $(\mathcal{P}, \delta_m)$ satisfies all the conditions of

Definition 2.2. Therefore, we can construct a crossed product $\mathcal{P}\times_{\delta_m}\mathbb{Z}_+$, which is a universal unital C^*-algebra generated by $\mathcal{P}$ and T^m.

The main purpose of this paper is to show that

$$\mathcal{P}\times_{\delta_m}\mathbb{Z}_+=\mathcal{T}_m.$$

Let us consider a triple $(\pi,l^2(\mathbb{Z}_+)),T^m)$, where $\pi\colon\mathcal{P}\to B(l^2(\mathbb{Z}_+))$ is a representation, which acts identically: $\pi(a)=a, a\in\mathcal{P}$. It is evident that π is a faithful representation. Obviously $(\pi,l^2(\mathbb{Z}_+)),T^m)$ satisfies the conditions (2.4), (2.5). Consider the C^*-algebra $C^*(\pi(\mathcal{P}),T^m)=C^*(\mathcal{P},T^m)$ generated by an algebra $\pi(\mathcal{P})=\mathcal{P}$ and T^m. Using the fact, that $(\mathcal{P},T^m)$ satisfies the conditions (2.4), (2.5), we obtain that $\mathcal{P}$ is a coefficient algebra for the C^*-algebra $C^*(\pi(\mathcal{P}),T^m)$. Denote by C_0 a vector space consisting of finite sums of the form (2,6), where $a_k\in\mathcal{P}$.

Lemma 4.3. C^*-algebra $C^*(\mathcal{P},T^m)$ generated by algebra $\mathcal{P}$ and by the operator T^m coincides with $\mathcal{T}_m$.

Proof. Observe that finite linear combination of monomial is dense in $\mathcal{T}_m$ and C_0 is dense in $C^*(\mathcal{P},T^m)$ and, therefore, in order to prove the Lemma we need to show that we can obtain any element from C_0 by the monomials, and conversely, any element of C_0 can be obtained by monomials. Since $\mathcal{P}$ is the coefficient algebra for $C^*(\mathcal{P},T^m)$, then we can represent $x\in C_0$ in the following way:

$$x=(T^m)^{*N}a_{-N}+\ldots+(T^m)^*a_{-1}+a_0+a_1T^m+\ldots+a_N(T^m)^N,\;\; a_k,a_{-k}\in\mathcal{P}. \tag{4.3}$$

In this decomposition an index of each summand is divisible by m. Indeed,

$$ind((T^m)^{*k}a_{-k})=ind((T^m)^{*k})+ind(a_{-k})=ind((T^m)^{*k})=mk,$$

as the index of each element from $\mathcal{P}$ is equal to 0. Now we show the opposite part. Let V be any monomial from $\mathcal{T}_m$: $V=T^{mk+i}T^{*i}, 1\le i<m$. Then V can be represented as: $V=(T^m)^kT^iT^{*i}$. If in the decomposition (2.7) we take $a_0=T^iT^{*i}$ and $a_k=0,\ k\neq 0$, we obtain $T^iT^{*i}\in B_0$. Similarly, in (2.7) taking $a_k=I$ and $a_j=0,\ j\neq k$, we obtain $a_k(T^m)^k=T^{mk}\in B_0$. Since B_0 is an algebra, we get $T^{mk}T^iT^{*i}=V\in B_0$. $\square$

We will show, that algebra $C^*(\mathcal{P},T^m)=\mathcal{T}_m$ possesses property (*). Denote by $L_{km}=\mathcal{P}T^{km},\ L_{-km}=T^{*km}\mathcal{P}$.

Lemma 4.4. The C^*-algebra $\mathcal{T}_m$ possesses property (*), that is

$$\| a_0 \|\le\| x \|,$$

where a_0 and x are the elements of the decomposition (4.3).

Proof. Let $C(S^1;\mathcal{T}_m)=C(S^1)\otimes\mathcal{T}_m$ be the C^*-algebra of all continuous functions on the unit circle S^1 with values in the algebra $\mathcal{T}_m$, with the uniform norm

$$\| b \|=\sup_{S^1}\| b(z) \|,\ b\in C(S^1;\mathcal{T}_m).$$

Every $\mathcal{T}_m$-valued function $b\in C(S^1;\mathcal{T}_m)$ can be written in a formal Fourier series:

$$b(z)\simeq\sum_{k=-\infty}^{\infty}b_kz^k,\;\text{ where }\; b_k=\frac{1}{2\pi}\int_0^{2\pi}b(e^{i\theta})e^{-ik\theta}d(\theta)\in\mathcal{T}_m. \tag{4.4}$$

Any element $b \in C(S^1;\mathfrak{T}_m)$ can be approximated by the norm of the algebra $C(S^1;\mathfrak{T}_m)$ by finite linear combination of the form (4.4).

For each monomial V of $\mathfrak{T}_m$ an action $\sigma_0\colon S^1 \to \mathrm{Aut}(\mathfrak{T}_m)$ defines a function $\tilde{V} \in C(S^1;\mathfrak{T}_m)$ as follows:

$$\tilde{V}(z) = \sigma_0(z)(V) = z^{\mathrm{ind}(V)}V.$$

Denote by $\tilde{\mathfrak{T}}_m$ the closed subalgebra of $C(S^1;\mathfrak{T}_m)$ generated by the functions $\tilde{V}$. Then, as in (4.4), each element $\tilde{a}(z) \in \tilde{\mathfrak{T}}_m \subset C(S^1;\mathfrak{T}_m)$ can be represented in a formal series:

$$\tilde{a}(z) \simeq \sum_{k=-\infty}^{\infty} a_k z^{km}, \quad \text{where } a_k = \frac{1}{2\pi}\int_0^{2\pi} \tilde{a}(e^{i\theta})e^{-ikm\theta}d(\theta) \in \mathfrak{T}_m. \tag{4.5}$$

As in the proof of Lemma 2.1 [7], it can be shown that the Fourier coefficients a_k of (4.5) lie in the corresponding subspaces L_{km}. Since $\tilde{a}(z) = \sigma(z)(a)$, where $a \in \mathfrak{T}_m$, it follows that $a \simeq \sum\limits_{k=-\infty}^{infty} a_k$, where $a_k \in L_{km}$. The algebras $\mathfrak{T}_m$ and $\tilde{\mathfrak{T}}_m$ are isometrically isomorphic [7] and, therefore, for each $a \in \mathfrak{T}_m$ the following inequality holds:

$$\| a_0 \| = \left\| \frac{1}{2\pi}\int_0^{2\pi} \tilde{a}(e^{i\theta})d(\theta) \right\| \leq \| \tilde{a}(z) \| = \| a \|$$

Thus, property (*) holds. □

Remark 4.1. Property (*) provides the uniqueness of the coefficients a_k, a_{-k} in the decomposition (4.3) and, consequently, the uniqueness of the decomposition (4.3).

Theorem 4.1. For every $m \in \mathbb{N}$ C^*-algebras $\mathfrak{T}_m$ and $\mathcal{P} \times_{\delta_m} \mathbb{Z}_+$ are isomorphic

$$\mathfrak{T}_m = C^*(\mathcal{P}, T^m) \cong \mathcal{P} \times_{\delta_m} \mathbb{Z} = \varphi(\mathcal{A}) \times_{\delta_m} \mathbb{Z}, \tag{4.6}$$

where the isomorphism

$$\psi\colon C^*(\mathcal{P}, T^m) \to \mathcal{P} \times_{\delta_m} \mathbb{Z}$$

is such that $\psi(T^m) = \hat{U}, \psi(a) = \hat{a}$, for all $a \in \mathcal{P}$, where $\hat{U}$ and $\hat{a}$ are the canonical images of $a \in \mathcal{P}$ and T^m in $\mathcal{P} \times_{\delta_m} \mathbb{Z}$, respectively, and δ_m, δ_{*m} are defined by formulas (4.1) and (4.2).

Proof. The equality $\mathfrak{T}_m = C^*(\mathcal{P}, T^m)$ is proved in Lemma 4.3. We prove in Lemma 4.4 that the algebra $C^*(\mathcal{P}, T^m)$ possesses the property (*). Besides, for the pair $(\mathcal{P}, \delta_m)$ due to Lemma 4.2, there exists full transfer operator δ_{*m}, so, using the Theorem 3.5 from [6], we obtain

$$C^*(\mathcal{P}, T^m) \cong \mathcal{P} \times_{\delta_m} \mathbb{Z}.$$

The last equality follows from Lemma 4.1. □

Corollary 4.1 (Coburn's theorem [8]). If we take $m = 1$ in (4.4), we obtain the representation of the Toeplitz algebra as a crossed product:

$$\mathcal{T} = C^*(\mathcal{P}, T) \cong \mathcal{P} \times_{\delta_1} \mathbb{Z} = \varphi(\mathcal{A}) \times_{\delta_1} \mathbb{Z}.$$

The author would like to thank Professor Grigorian S.A. for valuable advices and useful discussions.

Received 25.07.2014

REFERENCES

1. **Murphy G.J.** The Crossed Product of a C^*-Algebra by an Endomorphism. // Integral Equations Operator Theory, 1996, v. 24, № 3, p. 298–319.
2. **Paschke W.L.** The Crossed Product of a C^*-Algebra by an Endomorphism. // Proc. Amer. Math. Soc., 1980, v. 80, № 1, p. 113–118.
3. **Stacey P.J.** Crossed Products of C^*-Algebras by Endomorphisms. // J. Austral. Math. Soc. Ser. A, 1993, v. 54, № 2, p. 204–212.
4. **Exel R.** A New Look at the Crossed-Product of a C^*-Algebra by an Endomorphism. // Ergodic Theory Dynam. Systems, 2003, v. 23, № 6, p. 1733–1750.
5. **Lebedev A.V., Odzijewicz A.** Extensions of C*-Algebras by Partial Isometries. // Matem. Sbor., 2004, v. 195, p. 951–982.
6. **Antonevich A.B., Bakhtin V.I., Lebedev A.V.** Crossed Product of a C*-Algebra by an Endomorphism, Coefficient Algebras and Transfer Operators. // Matem. Sbor., 2011, v. 202, p. 1253–1283.
7. **Aukhadiev M.A., Grigorian S.A., Lipacheva E.V.** Infinite-Dimensional Compact Quantum Semigroup. // Lobachevskii Journal of Mathematics, 2011, v. 32, № 4, p. 304–316.
8. **Coburn L.A.** The C*-Algebra Generated by an Isometry. // Bull. Amer. Math. Soc., 1967, № 73, p. 722–726.