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*M a t h e m a t i c s*

# ON SOLUTION OF A CLASS OF HAMMERSTEIN TYPE NONLINEAR INTEGRAL EQUATIONS ON THE POSITIVE HALF-LINE IN THE CRITICAL CASE

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In the this article a class of nonlinear integral equations with noncompact Hammerstein integral operator, the kernel of which depends on difference of its arguments is investigated. Above mentioned class of equations arises in the kinetic theory of gases and in the radiative transfer theory in nuclear reaction.

Combination of special iteration methods with the methods of the theory of construction of invariant cone-shaped segments allow to prove existence theorems of positive solutions in special selected weighted space.

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**Introduction.** The work is devoted to the study of the following class of Hammerstein type nonlinear integral equations:

$$f(x)=\mu_0(x,f(x))+\int_0^\infty K(x-t)\mu_1(t,f(t))dt, \quad x\in\mathbb{R}^+\equiv[0,+\infty), \tag{1}$$

with respect to unknown real and measurable function $f(x)$. Here the kernel $K(\tau)$, defined on $(-\infty,+\infty)$, is an even and nonnegative measurable function, satisfying the conservative or supercritical conditions:

$$K(\tau)\geq 0, \quad \tau\in\mathbb{R}, \quad K\in L_1(\mathbb{R}), \alpha\equiv\int_{-\infty}^{+\infty}K(\tau)d\tau\geq 1. \tag{2}$$

The case $\alpha=1$ corresponds to the condition of conservativity and the case $\alpha>1$ is the supercritical condition.

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The functions $\{\mu_j(x,u)\}_{j=0,1}$ are defined on the set $\mathbb{R}^+ \times \mathbb{R}$, take real values and satisfy the critical condition

$$\mu_j(x,0) \equiv 0, \quad \forall x \in \mathbb{R}^+, \quad j = 0,1. \tag{3}$$

This condition is caused by the fact that the identically zero function is a solution of (1). The main aim of the present work is the construction of a nontrivial nonnegative (physical) solution of (1). The Eq. (1), other than purely theoretical interest, has applications in the kinetic theory of gases (in the temperature-jump nonlinear problem). In the case when $\mu_0 \equiv 0$ and the kernel is a completely monotonic function, the Eq. (1) has also applications in the nonlinear theory of radiative transfer in nuclear reactors [1–3].

In the case when $\mu_0 \equiv 0$, depending on the properties of the function $\mu_1(t,z)$, in [5–8], it is discussed the global solvability of (1) in the space of essentially bounded functions or in the space of measurable functions that possesses a linear growth at infinity. In a recent paper of Kh.A. Khachatryan (see. [9]) the Eq. (1) is investigated in the case $\alpha = 1, \nu(K) = \int\limits_{-\infty}^{+\infty} xK(x)dx < 0$, for functions $\{\mu_j(x,z)\}_{j=0,1}$ with the Caratheodory's condition [10], monotonic in the second argument and satisfying the following inequalities:
there exist positive numbers $\eta > 0$ and $\eta_0 \in (0,\eta)$ such that

$$\mu_0(x,\Phi_{\eta_0}(x)) \geq \Phi_{\eta_0}(x), \quad \mu_0(x,\eta) \leq \Phi_\eta(x), \quad x \in \mathbb{R}^+,$$
$$0 \leq \mu_1(t,z) \leq z, \quad t \in \mathbb{R}^+, \quad z \in [0,\eta],$$

where

$$\Phi_\delta(x) \equiv \delta \int\limits_x^\infty K(u)du, \quad x \in \mathbb{R}^+, \ \delta > 0.$$

In this paper, using the methods of the theory of constructing invariant cone segments for the corresponding nonlinear monotonic operator, by means of special iterative methods and some a'priori estimates, we prove the theorem constructive global solvability of (1) in a certain weighted space. At the end of the work specific examples of functions $\{\mu_j(x,z)\}_{j=0,1}$ are listed, for which the conditions of the theorems are fulfilled.

**Construction a Positive Solution of the Eq. (1) in the Conservative Case.** The following Theorem is true

***Theorem 1.*** Suppose that there exists an integrable function $\lambda(x)$ on $\mathbb{R}^+$

$$0 \leq \lambda(x) \leq 1, x \in \mathbb{R}^+, \lambda(x) \not\equiv 0,$$

such that:

$i_1$) $\mu_0(x,\lambda(x)) \geq \lambda(x)$, $x \in \mathbb{R}^+$, $0 \leq \mu_1(x,u) \leq u$, $x \in \mathbb{R}^+$, $u \geq \lambda(x)$;

$i_2$) for each fixed $x \in \mathbb{R}^+$ the functions $\{\mu_j(x,u)\}_{j=0,1}$ are monotonically increasing with respect to $u$ on $[\lambda(x), +\infty)$;

$i_3)$ the functions $\{\mu_j(x,u)\}_{j=0,1}$ satisfy the Caratheodory's condition with respect to the argument $u$ on the set $\mathbb{R}^+ \times \mathbb{R}^+$, i.e. for each fixed $u \in \mathbb{R}^+$ the functions $\{\mu_j(x,u)\}_{j=0,1}$ are measurable in $x$ and for almost all $x \in \mathbb{R}^+$ the functions are continuous in $u \in \mathbb{R}^+$;

$i_4)$ the following inequality holds:

$$\int_0^\infty \operatorname*{ess\,sup}_{z\geq 0} \mu_0(x,z)dx \equiv c < +\infty. \tag{4}$$

Then, if $\alpha = 1$, the Eq. (1) has a nonnegative solution $f(x)$ from the following weighted space:

$$\mathfrak{M}_\gamma \equiv \left\{\varphi(x), x \in \mathbb{R}^+ : \varphi(x) \text{ is measurable in } \mathbb{R}^+ \text{ and } \int_0^\infty \gamma(x)|\varphi(x)|dx < +\infty\right\},$$

where

$$\gamma(x) \equiv \int_x^\infty K(u)du \not\equiv 0, \quad x \geq 0. \tag{5}$$

*P r o o f*. Consider the following special iteration:

$$\begin{gathered} f_{n+1}(x) = \mu_0(x, f_n(x)) + \int_0^\infty K(x-t)\mu_1(t, f_n(t))dt, \quad x \in \mathbb{R}^+, \\ f_0(x) = \lambda(x), \quad n = 0,1,2,\ldots \end{gathered} \tag{6}$$

Applying induction method on $n$, we will verify that

$$a) \quad f_n(x) \uparrow \text{ with respect to } n, \tag{7}$$

$$b) \quad f_n \in L_1(\mathbb{R}^+),\ n = 0,1,2,\ldots \tag{8}$$

First we will prove the assertion $a)$.

Inequality $f_1(x) \geq f_0(x)$ follows directly from the following inequalities:

$$f_1(x) \geq \mu_0(x, f_0(x)) = \mu_0(x, \lambda(x)) \geq \lambda(x) = f_0(x).$$

Suppose that $f_n(x) \geq f_{n-1}(x)$ for some $n \in \mathbb{N}$, using the monotonicity of the functions $\{\mu_j(x,u)\}_{j=0,1}$ with respect to $u$ on $[\lambda(x), +\infty)$, from (6) we obtain

$$f_{n+1}(x) \geq \mu_0(x, f_{n-1}(x)) + \int_0^\infty K(x-t)\mu_1(t, f_{n-1}(t))dt = f_n(x).$$

Now let prove the assertion $b)$.

In the case $n = 0$ the inclusion $b)$ follows directly from the conditions of the Theorem 1. Let $b)$ holds for some $n \in \mathbb{N}$. Then, taking into consideration the conditions $i_1)$, $i_2)$, $i_4)$, from (6) for any $r > 0$, we have

$$\int_0^r f_{n+1}(x)dx \leq \int_0^r \mu_0(x, f_n(x))dx + \int_0^r \int_0^\infty K(x-t)f_n(t)dtdx \leq$$

$$\leq \int_0^\infty \mu_0(x, f_n(x))dx + \int_0^\infty f_n(t) \int_0^r K(x-t)dxdt \leq c + \int_0^\infty f_n(t)dt < +\infty,$$

because of $\int_{-\infty}^{+\infty} K(u)du = 1$ and $K(x) \geq 0, x \in \mathbb{R}^+$.

From this inequality, tending $r \to \infty$, we conclude that

$$\int_0^\infty f_{n+1}(x)dx < +\infty.$$

Integrating both sides of (6) with respect to $x$ from 0 to $+\infty$, from conditions $K(-u) = K(u), u \geq 0,$ and $i_1),\ i_2),\ i_4)$ we obtain

$$\int_0^\infty f_{n+1}(x)dx \leq c + \int_0^\infty f_n(t) \int_0^\infty K(x-t)dxdt \leq c + \int_0^\infty f_{n+1}(t) \int_{-t}^\infty K(u)dudt =$$

$$= c + \int_0^\infty f_{n+1}(t) \int_{-\infty}^t K(-u)dudt = c + \int_0^\infty f_{n+1}(t)(1-\gamma(t))dt,$$

and, therefore, we have

$$\int_0^\infty f_{n+1}(t)\gamma(t)dt \leq c. \tag{9}$$

Consequently, taking into account (7) and (9) from the theorem of B. Levi (see. [11]), it follows that the sequence of functions $\left\{f_n(x)\right\}_{n=0}^\infty$ has a pointwise limit when $n \to \infty : \lim\limits_{n\to\infty} f_n(x) = f(x)$ almost everywhere on $\mathbb{R}^+$.

We prove that the limit function $f$ satisfies the Eq. (1). Indeed, as the sequence of measurable functions $\{f_n(x)\}_{n=0}^\infty$ are monotonically nondecreasing with respect to $n$, and $\{\mu_j(x,u)\}_{j=0;1}$ are also monotonic in $u$, then due to the nonnegativity of the kernel $K$ and the Caratheodory's condition, we conclude that the functions

$$F_n(x,t) = K(x-t)\mu_1(t, f_n(t)),\ n = 0, 1, \ldots, \infty,$$

are measurable with respect to $t$ for each fixed $x \in \mathbb{R}^+$ and monotonically nondecreasing with respect to $n$.

On the other hand, from (2), $i_1)$ and $i_2)$ it follows that $\int_0^\infty F_n(x,t)dt \leq f(x)$. Hence, according to B. Levi theorem, the sequence of functions $F_n(x,t)_{n=0}^\infty$ for any fixed $x \in \mathbb{R}^+$ has pointwise limit $\lim\limits_{n\to\infty} F_n(x,t) = F(x,t)$ almost everywhere on $\mathbb{R}^+$ and

$$\lim_{n\to\infty} \int_0^\infty F_n(x,t)dt = \int_0^\infty \lim_{n\to\infty} F_n(x,t)dt \leq f(x).$$

Since, the functions $\mu_j(x,u)_{j=0,1}$ satisfy the Caratheodory's conditions and $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere on $\mathbb{R}^+$, from Krasnoselskii theorem (see [12]) follows that $\lim_{n\to\infty} \mu_j(x, f_n(x)) = \mu_j(x, f(x)),\ j = 0,1$, almost everywhere on $\mathbb{R}^+$.

Hence, the limit function satisfies the Eq. (1).

Tending $n\to\infty$ in (9) and using the fact that $f_n(x)\uparrow$ with respect to $n$, from (7) and (9) we obtain

$$f(x) \geq \lambda(x),\, x\in\mathbb{R}^+, \quad \int_0^\infty f(x)\gamma(x)dx \leq c, \tag{10}$$

i.e $f\in\mathfrak{M}_\gamma$. □

**On the Solvability of Eq. (1) in the Case, when the Kernel $K$ Satisfies Supercritical Conditions and is Completely Monotonic Function.** In this section we construct a positive solution of the Eq. (1) in the case when the kernel has the following representation:

$$K(\tau) = \int_a^b e^{-|\tau|s} d\sigma(s), \quad \tau\in\mathbb{R}, \tag{11}$$

and satisfies the supercriticality condition

$$\alpha \equiv \int_{-\infty}^{+\infty} K(\tau)d\tau = 2\int_a^b \frac{1}{s} d\sigma(s) > 1, \tag{12}$$

where $\sigma(s)$ is a nondecreasing and continuous function on $[a,b)$ and $0<a<b\leq+\infty$.

In the case when $\mu_0\equiv 0$ and the kernel allow the representation (11) the Eq. (1) arises in nonlinear transfer theory of nuclear reactors.

We introduce the following function:

$$K_\varepsilon(x) = K(x)e^{\varepsilon x}, \quad \varepsilon\in[0,a), \quad x\in\mathbb{R}. \tag{13}$$

From the representation (11) of the kernel $K$ it immediately follows that

$$K_\varepsilon(x) = \begin{cases} \int_a^b e^{-x(s-\varepsilon)} d\sigma(s), & x\geq 0, \\ \int_a^b e^{x(s+\varepsilon)} d\sigma(s), & x<0, \end{cases} \in L_1(-\infty,+\infty), \tag{14}$$

due to $\varepsilon\in[0,a)$ and $s\geq a$.

Consider the function:

$$\rho(\varepsilon) \equiv \int_{-\infty}^0 K_\varepsilon(\tau)d\tau = \int_a^b \frac{1}{s+\varepsilon} d\sigma(s), \quad \varepsilon\in[0,a). \tag{15}$$

We note that

$$\rho\in C[0,a), \quad \rho(0) = \frac{\alpha}{2} > \frac{1}{2},\, \rho(\varepsilon)\downarrow \text{ with respect to } \varepsilon \text{ on } [0,a). \tag{16}$$

Hence, by Cauchy's theorem there exists a number $\varepsilon_0\in(0,a)$, such that $\rho(\varepsilon_0)\geq \frac{1}{2}$.

We fix a number $\varepsilon_0$. The following theorem is valid:

***Theorem 2.*** Let the functions $\{\mu_j(t,u)\}_{j=0,1}$ satisfy the following conditions:

$j_1)$ for any fixed $t\in\mathbb{R}^+$ the functions $\{\mu_j(t,u)\}_{j=0,1}$ are monotonically increasing with respect to $u$ on $[e^{-at},+\infty)$;

$j_2)$ the following inequalities are true:

$$\mu_0(t,u)\geq 0,\, t\in\mathbb{R}^+,\, u\geq e^{-at},\, c\equiv\int\limits_0^\infty \operatorname*{esssup}_{u\geq 0}\mu_0(t,u)dt<+\infty,$$

$$\mu_1(t,e^{-\varepsilon_0 t})\geq 2e^{-\varepsilon_0 t},\ \mu_1(t,u)\leq\frac{u}{\alpha}+\beta_{\varepsilon_0}(t),\ t\in\mathbb{R}^+,\ u\geq e^{-at},$$

for some integrable functions $\beta_{\varepsilon_0}(t)$ on $\mathbb{R}^+:\beta_{\varepsilon_0}(t)\geq 2e^{-\varepsilon_0 t},\, t\in\mathbb{R}^+$;

$j_3)$ the functions $\{\mu_j(t,u)\}_{j=0,1}$ satisfy the Caratheodory's condition with respect to the argument $u$ on $\mathbb{R}^+\times\mathbb{R}^+$.

Then, if $\alpha>1$, the Eq. (1) has a positive solution in the space $\mathfrak{M}_\gamma$.

*Proof*. Consider the following iteration:

$$\begin{gathered} f_{n+1}(x)=\mu_0(x,f_n(x))+\int\limits_0^\infty K(x-t)\mu_1(t,f_n(t))dt,\quad x\geq 0,\\ f_0(x)=e^{-\varepsilon_0 x},\quad n=0,1,2,\ldots \end{gathered}\tag{17}$$

Applying the induction method on $n$, first make sure that

$$f_n(x)\uparrow \text{ with respect to } n.\tag{18}$$

Indeed, in view of conditions $j_2)$ and $j_1)$ from (17) we obtain for $x\geq 0$

$$f_1(x)\geq\int\limits_0^\infty K(x-t)\mu_1(t,f_0(t))dt=\int\limits_0^\infty K_{\varepsilon_0}(x-t)e^{-\varepsilon_0(x-t)}\mu_1(t,e^{-\varepsilon_0 t})dt\geq$$

$$\geq 2\int\limits_0^\infty K_{\varepsilon_0}(x-t)e^{-\varepsilon_0(x-t)}e^{-\varepsilon_0 t}dt=2e^{-\varepsilon_0 x}\int\limits_{-\infty}^x K_{\varepsilon_0}(u)du\geq$$

$$\geq 2e^{-\varepsilon_0 x}\rho(\varepsilon_0)\geq e^{-\varepsilon_0 x}=f_0(x),$$

because of $\rho(\varepsilon_0)\geq\dfrac{1}{2}$.

Assuming that $f_n(x)\geq f_{n-1}(x)$ for some $n\in\mathbb{N}$ from the condition $j_1)$ it follows that

$$f_{n+1}(x)\geq\mu_0(x,f_{n-1}(x))+\int\limits_0^\infty K(x-t)\mu_1(t,f_{n-1}(t))dt=f_n(x).$$

Thus the monotonicity of the sequence $\Big\{f_n(x)\Big\}_{n=0}^\infty$ with respect to $n$ is proved. Similarly, as in the proof of Theorem 1, by the induction in $n$ we can see that $f_n\in L_1(\mathbb{R}^+)$.

Integrating both sides of (17) with respect to $x$ from 0 to $+\infty$, using the inequalities $j_2)$ and the monotonicity condition, we obtain the following chain of inequalities:

$$\int_0^\infty f_{n+1}(x)dx \leq c + \int_0^\infty \int_0^\infty K(x-t)\mu_1(t, f_n(t))dtdx \leq$$

$$\leq c + \int_0^\infty \int_0^\infty K(x-t)\mu_1(t, f_{n+1}(t))dtdx \leq$$

$$\leq c + \int_0^\infty \int_0^\infty K(x-t)\left(\frac{f_{n+1}(t)}{\alpha} + \beta_{\varepsilon_0}(t)\right)dtdx \leq$$

$$\leq c + \frac{1}{\alpha}\int_0^\infty f_{n+1}(t)\int_{-t}^\infty K(u)dudt + \alpha\int_0^\infty \beta_{\varepsilon_0}(t)dt =$$

$$= c + \alpha\int_0^\infty \beta_{\varepsilon_0}(t)dt + \frac{1}{\alpha}\int_0^\infty f_{n+1}(t)\int_{-\infty}^t K(u)dudt =$$

$$= c + \alpha\int_0^\infty \beta_{\varepsilon_0}(t)dt + \frac{1}{\alpha}\int_0^\infty f_{n+1}(t)(\alpha - \gamma(t))dt.$$

From this inequality it immediately follows that

$$\int_0^\infty f_{n+1}(t)\gamma(t)dt \leq c\alpha + \alpha^2\int_0^\infty \beta_{\varepsilon_0}(t)dt < +\infty. \tag{19}$$

Thus given (18) and (19) conclude that the sequence of functions $\{f_n(x)\}_{n=0}^\infty$ has a limit when $n \to \infty : \lim\limits_{n\to\infty} f_n(x) = f(x)$, and the limit function $f(x)$ satisfies the Eq. (1) and the following inequalities:

$$f(x) \geq e^{-\varepsilon_0 x}, \qquad \int_0^\infty f(x)\gamma(x)dx \leq c\alpha + \alpha^2\int_0^\infty \beta_{\varepsilon_0}(t)dt, \qquad x \in \mathbb{R}^+. \qquad \square$$

At the end we give some examples of functions $\{\mu_j(t,u)\}_{j=0,1}$, for which the conditions of the formulated theorems are fullfiled.

**Examples of Functions $\{\mu_j(t,u)\}_{j=0,1}$ for Theorem 2.**

$c)$ $\mu_0(t,u) = (1-e^{-u})e^{-t^2}, \quad u \geq 0, \quad t \in \mathbb{R}^+;$

$d)$ $\mu_1(t,u) = \xi\sqrt{e^{-\varepsilon_0 t}u}, \qquad u \geq e^{-\varepsilon_0 t}, \quad \xi \geq \max\left\{2, \sqrt{\frac{8}{\alpha}}\right\}$ arbitrary number,

$\beta_{\varepsilon_0}(t) \geq \frac{\xi^2\alpha}{4}e^{-\varepsilon_0 t}, t \in \mathbb{R}^+, \ \beta_{\varepsilon_0} \in L_1(\mathbb{R}^+).$

Performance of the corresponding conditions of Theorems 1 and 2 for above mentioned examples $a), b), c)$ can be easily verified. Let us consider the example $d)$.

First note that $\mu_1(t,u) \uparrow$ with respect to $u$ for each fixed $t \in \mathbb{R}^+$. It is obvious that

$$\mu_1(t,u) \geq \xi\sqrt{e^{-2\varepsilon_0 t}} = \xi e^{-\varepsilon_0 t} \geq 2e^{-\varepsilon_0 t}, \text{ because of } \xi \geq 2.$$

The function $\mu_1(t,u) = \xi\sqrt{e^{-\varepsilon_0 t}u}$ is jointly continuous in its arguments on set $\mathbb{R}^+ \times \mathbb{R}^+$, hence, it satisfies the Caratheodory's condition.

Below we prove that there exists an integrable function $\beta_{\varepsilon_0}(t)$ on $\mathbb{R}^+$ : $\beta_{\varepsilon_0}(t) \geq \frac{\xi^2\alpha}{4}e^{-\varepsilon_0 t}$, $t \in \mathbb{R}^+$, such that

$$\xi\sqrt{e^{-\varepsilon_0 t}u} \leq \frac{u}{\alpha} + \beta_{\varepsilon_0}(t),\ t \in \mathbb{R}^+,\ u \geq 0. \tag{20}$$

Indeed, this inequality is equivalent to the following inequality:

$$u^2 + [2\alpha\beta_{\varepsilon_0}(t) - \xi^2\alpha^2 e^{-\varepsilon_0 t}]u + \alpha^2\beta_{\varepsilon_0}^2(t) \geq 0, \quad t \in \mathbb{R}^+,\ u \geq 0. \tag{21}$$

It is obvious that the inequality (21) takes place, if

$$[2\alpha\beta_{\varepsilon_0}(t) - \xi^2\alpha^2 e^{-\varepsilon_0 t}]^2 - 4\alpha^2\beta_{\varepsilon_0}^2(t) \leq 0,$$

which is equivalent to the following inequality:

$$\beta_{\varepsilon_0}(t) \geq \frac{\xi^2\alpha}{4}e^{-\varepsilon_0 t}, \quad t \in \mathbb{R}^+.$$

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