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*Mathematics*

# PAIR OF LINES AND MAXIMAL PROBABILITY

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In this paper we consider two independent and identically distributed lines, which intersect a planar convex domain **D**. We evaluate the probability $P_{\mathbf{D}}$, for the lines to intersect inside **D**.

Translation invariant measures generating random lines is obtained, under which $P_{\mathbf{D}}$ achieves its maximum for a disc and a rectangle. It is also shown that for every $p$ from the interval $[0,1/2]$ and for every square there are measures generating random lines such that $P_{\mathbf{D}} = p$.

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**Introduction.** Consider a pair of random lines generated by a translation invariant measure, which intersect planar convex domain **D**. The probability $P_{\mathbf{D}}$ that intersection of two lines belongs **D** depends on the measure generating random lines. The explicit form of the mentioned probability for the Euclidian measure is given in [1] for every **D**. We are interested in maximizing probability $P_{\mathbf{D}}$.

Let $G$ be the space of all lines $g$ in the Euclidean plane $\mathbb{R}^2$, and let $(p,\varphi)$ be the polar coordinates of the foot of the perpendicular to $g$ from the origin $O$. Let $\mu$ be a translation invariant measure on the space $G$. It is well known that the element of translation invariant measure up to a constant factor has the following form (see [2]):

$$\mu(dg) = dpm(d\varphi),$$

where $dp$ is the one-dimensional Lebesgue measure and $m$ is a finite measure on $S^1$ ($S^1$ is the circle of radius 1 centered at the origin).

For each bounded convex domain **D**, denote the set of lines that intersect **D** by

$$[\mathbf{D}] = \{g \in G,\ g \cap \mathbf{D} \neq \emptyset\}.$$

Consider two independent and identically distributed lines $g_1$ and $g_2$ meeting **D**. Denote by $A$ the set of all pairs $(g_1,g_2)$ intersecting inside **D**:

$$A = \{(g_1,g_2) \in [\mathbf{D}] \times [\mathbf{D}] : g_1 \cap g_2 \in \mathbf{D}\}.$$

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We assume that the lines are distributed by translation invariant measure $\mu$. Let $P_{\mathbf{D}} = P_{\mathbf{D}}(A)$ is the probability of $A$, defined by the following way:

$$P_{\mathbf{D}}(A) = (\mu \times \mu)(A)/\mu^2([\mathbf{D}]). \tag{1}$$

**Calculation of** $(\mu \times \mu)(A)$ **and** $\mu([\mathbf{D}])$. Let calculate the translation invariant measure of $[\mathbf{D}]$ :

$$\mu([\mathbf{D}]) = \int\limits_{[\mathbf{D}]} \mu(dg) = \int_0^{2\pi} m(d\varphi) \int_0^{p(\varphi)} dp = \int_0^{2\pi} p(\varphi) m(d\varphi), \tag{2}$$

here $p(\varphi)$ is the support function of $\mathbf{D}$ (for the definition of support function see [1]).

In order to obtain $(\mu \times \mu)(A)$ the following integral is calculated:

$$(\mu \times \mu)(A) = \int\limits_A \mu(dg_1)\mu(dg_2) = \int_0^{2\pi} \int_0^{p(\varphi_1)} dp_1 m(d\varphi_1) \int\limits_{[\chi(p_1,\varphi_1)]} dp_2 m(d\varphi_2),$$

where $\chi(p_1, \varphi_1) = g_1 \cap \mathbf{D}$ (below by $\chi(p_1, \varphi_1)$ we denote both a chord and its length). If a line with direction $\varphi_2$ intersects $\chi(p_1, \varphi_1)$, its $p_2$ coordinate should be from the projection of $\chi(p_1, \varphi_1)$ on direction $\varphi_2$. The length of the projection is $\chi(p_1, \varphi_1) \cdot |\sin(\varphi_1 - \varphi_2)|$. Hence we have the general form for $(\mu \times \mu)(A)$ :

$$(\mu \times \mu)(A) = \int_0^{2\pi} \int_0^{p(\varphi_1)} dp_1 m(d\varphi_1) \int_0^{\pi} \chi(p_1, \varphi_1) |\sin(\varphi_1 - \varphi_2)| m(d\varphi_2) =$$

$$= \int_0^{2\pi} m(d\varphi_1) \int_0^{p(\varphi_1)} \chi(p_1, \varphi_1) dp_1 \int_0^{\pi} |\sin(\varphi_1 - \varphi_2)| m(d\varphi_2). \tag{3}$$

*The Case of Disk.*

When $\mathbf{D}$ is a disc of radius $R$, we can take the origin $O$ as the center of the disc. In this case we have $p(\varphi_1) = R$ for every direction $\varphi_1$, hence we can easily calculate (2):

$$\mu([\mathbf{D}]) = Rm(S^1). \tag{4}$$

Furthermore, in the disc case $\chi(p_1, \varphi_1)$ does not depend on $\varphi_1$ and $\chi(p_1, \varphi_1) = 2\sqrt{R^2 - p_1^2}$. It is easy to calculate, that

$$\int_0^R \sqrt{R^2 - p_1^2}\, dp_1 = \frac{\pi R^2}{2}. \tag{5}$$

Thus, substituting (5) in (3), we obtain:

$$(\mu \times \mu)(A) = \frac{\pi R^2}{2} \int_0^{2\pi} m(d\varphi_1) \int_0^{\pi} |\sin(\varphi_1 - \varphi_2)| m(d\varphi_2). \tag{6}$$

Using (4) and (6), from (1) we obtain the following expression for $P_{\mathbf{D}}(A)$, when $\mathbf{D}$ is a disc of radius $R$:

$$P_{\mathbf{D}}(A) = \frac{1}{R^2 m^2(S^1)} \cdot \frac{\pi R^2}{2} \int_0^{2\pi} m(d\varphi_1) \int_0^{\pi} |\sin(\varphi_1 - \varphi_2)| m(d\varphi_2) =$$

$$= \frac{\pi}{m^2(S^1)} \int_0^{\pi} m(d\varphi_1) \int_0^{\pi} |\sin(\varphi_1 - \varphi_2)| m(d\varphi_2).$$

We can estimate integrals in the expression of $P_{\mathbf{D}}(A)$ by the constant factor $\frac{m^2(S^1)}{4}\cdot\frac{2}{\pi}$ (see [2]), hence we obtain:

$$P_{\mathbf{D}}(A)\leqslant\frac{\pi}{m^2(S^1)}\cdot\frac{m^2(S^1)}{2\pi}=\frac{1}{2}.$$

It is known, that for the isotropic measure $\mu=dpd\varphi$, the probability $P_{\mathbf{D}}(A)=\frac{1}{2}$, hence, for the Euclidian invariant measure the probability for two random lines meeting **D** intersect inside **D** achieves its maximum.

The same result is also shown in [3].

*The Case of Rectangle.*

Suppose that **D** is a rectangle with sides $a$ and $b$ ($a\leqslant b$) and $m$ is a measure on $S^1$ concentrated only at two directions, perpendicular to the directions of rectangle sides. Denote these directions by $u_1$ perpendicular to $b$ and $u_2$ perpendicular to $a$. We can take the origin as the intersection point of the diagonals of the rectangle. Let us calculate $P_{\mathbf{D}}(A)$ in the mentioned case. From (2) we obtain:

$$\mu([\mathbf{D}])=\frac{a}{2}(m(u_1)+m(u_1+\pi))+\frac{b}{2}(m(u_2)+m(u_2+\pi))=am(u_1)+bm(u_2). \quad (7)$$

In order to calculate $(\mu\times\mu)(A)$ in this case, note that

$$\int_0^{\pi}|\sin(\varphi_1-\varphi_2)|m(d\varphi_2)=|\sin(\varphi_1-u_1)|m(u_1)+|\sin(\varphi_1-u_2)|m(u_2). \quad (8)$$

It is easy to see, that $p(u_1)=p(u_1+\pi)=\frac{a}{2}$ and $p(u_2)=p(u_2+\pi)=\frac{b}{2}$. For the directions $u_1$ and $u_2$ we have $\chi(p_1,u_1)=b$ and $\chi(p_1,u_2)=a$ respectively. Therefore, substituting (8) in (3), we obtain

$$(\mu\times\mu)(A)=(|\sin(u_1-u_1)|m(u_1)+|\sin(u_1-u_2)|m(u_2))\,abm(u_1)+$$
$$+(|\sin(u_2-u_1)|m(u_1)+|\sin(u_2-u_2)|m(u_2))\,bam(u_2).$$

Since $u_1$ and $u_2$ are perpendicular, we have $|\sin(u_1-u_2)|=1$. Thus we have

$$(\mu\times\mu)(A)=2abm(u_1)m(u_2). \quad (9)$$

Substituting (7) and (9) in (1), we obtain

$$P_{\mathbf{D}}(A)=\frac{2abm(u_1)m(u_2)}{(am(u_1)+bm(u_2))^2}. \quad (10)$$

It is easy to check, that if $m(u_1)=\frac{b}{a}m(u_2)$, then $P_{\mathbf{D}}(A)=\frac{1}{2}$.

It is proved in [3], that $P_{\mathbf{D}}(A)\leq\frac{1}{2}$ for every translation invariant measure $\mu$ and every bounded convex body in the plane. Thus, we describe a measure for which $P_{\mathbf{D}}(A)$ achieves its maximum for the rectangle **D**.

If **D** is a square with side $a$, then from (10) we have

$$P_{\mathbf{D}}(A)=\frac{2m(u_1)m(u_2)}{(m(u_1)+m(u_2))^2}. \quad (11)$$

After simple transformations of (11) we obtain the following equation:

$$P_{\mathbf{D}}(A)(m^2(u_1)+m^2(u_2)) = 2(1-P_{\mathbf{D}}(A))m(u_1)m(u_2). \tag{12}$$

Denoting $q = \dfrac{m(u_2)}{m(u_1)}$, we get:

$$P_{\mathbf{D}}(A)\,q^2 + 2q\,(P_{\mathbf{D}}(A)-1) + P_{\mathbf{D}}(A) = 0. \tag{13}$$

It is easy to check, that for every $p \leqslant \dfrac{1}{2}$ Eq. (13) has solutions $q = \dfrac{1-p\pm\sqrt{1-2p}}{p}$, therefore, there are $m(u_1)$ and $m(u_2)$ satisfying (11). Thus, for every probability $p$ not greater $1/2$ and for every square we construct a measure generating random lines such that $P_{\mathbf{D}}(A) = p$.

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