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M a t h e m a t i c s

A REMARK ON SYMMETRIC β-UNIFORMLY CLOSED ALGEBRA

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The conjugate space of β-uniformly closed, symmetric algebras of functions on topological spaces are investigated. An application of such spaces in algebras of almost periodic functions is given.

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Let Ω be a topological space and $C_b(\Omega)$ be closed with respect to the sup-norm an algebra of bounded complex-valued continuous functions on Ω. Here it is not supposed that the algebra $C_b(\Omega)$ separates the points of Ω. We denote by $\mathcal{A}_\infty(\Omega)$ a symmetric subalgebra of $C_b(\Omega)$ closed with respect to the sup-norm on Ω and containing the unit. By Theorem 2 (see [1], p. 251), we have $\mathcal{A}_\infty(\Omega) = C(M_{\mathcal{A}_\infty})$, where $M_{\mathcal{A}_\infty}$ is the space of maximal ideals of the algebra $\mathcal{A}_\infty(\Omega)$. Therefore, to the each point $x \in \Omega$ one can assign an element $\varphi \in M_{\mathcal{A}_\infty}$ such that for any function $f \in \mathcal{A}_\infty(\Omega)$ the equality $f(x) = \hat{f}(\varphi)$ holds, where $\hat{f} \in C(M_{\mathcal{A}_\infty})$ is a Gelfand transformation of f. The arising "contraction" mapping $\Phi : \Omega \to M_{\mathcal{A}_\infty}$, defined by the formula $\Phi(x) = \varphi$, where $f(x) = \hat{f}(\varphi)$ for all $f \in \mathcal{A}_\infty(\Omega)$ takes Ω into $M_{\mathcal{A}_\infty}$ and is continuous, since each neighborhood of $\varphi_0 \in M_{\mathcal{A}_\infty}$ with respect to the induced topology of $C(M_{\mathcal{A}_\infty})$ contains a neighborhood of the form

$$\big\{\varphi \in \Phi(\Omega) : \big|\hat{f}_k(\varphi) - \hat{f}_k(\varphi_0)\big| < \varepsilon;\ k = 1, \dots, n\big\} = \\ = \Phi\big(\{x \in \Omega : |f_k(x) - f_k(x_0)| < \varepsilon;\ k = 1, \dots, n\}\big),$$

where $\varphi_0 = \Phi(x_0)$. Thus, $\mathcal{A}_\infty(\Omega) = \{f \circ \Phi \in C_b(\Omega) : \hat{f} \in C(M_{\mathcal{A}_\infty})\}$. In view of the fact that for each function $f \in \mathcal{A}_\infty(\Omega)$, we have $f(x) = \hat{f}(\varphi) = \hat{f}(\Phi(x))$, where $\varphi \in M_{\mathcal{A}_\infty}$, the space $\Phi(\Omega)$ is dense in $M_{\mathcal{A}_\infty}$. Thus, since $M_{\mathcal{A}_\infty}$ is a compact Hausdorff space, we conclude that $M_{\mathcal{A}_\infty}$ is a completely regular space and therefore $M_{\mathcal{A}_\infty}$ is a compact extension of $\Phi(\Omega)$, that is $\Phi(\Omega)$ is a completely regular space.

Denoted by $\mathcal{B}(\Phi(\Omega))$ the algebra of all bounded complex-valued functions on $\Phi(\Omega)$. Let $\mathcal{B}_0(\Phi(\Omega))$ be the ideal of $\mathcal{B}(\Phi(\Omega))$ consisting of the functions vanishing

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in infinity (i.e. for each $\varepsilon > 0$ there exist a compact $K_\varepsilon \subset \Phi(\Omega)$ such that $|f(y)| < \varepsilon$ for $y \in \Phi(\Omega) \setminus K_\varepsilon$).

Using $\mathcal{B}_0(\Phi(\Omega))$, we define the family of seminorms $\{P_g\}_{g\in\mathcal{B}_0(\Phi(\Omega))}$ for the algebra $C_b(\Phi(\Omega))$ by $P_g(f) = \|T_g(f)\|_\infty$, where $T_g : C_b(\Phi(\Omega)) \to C_b(\Phi(\Omega))$ is the multiplicative operator $T_g(f) = fg$. The topology on the algebra $C_b(\Phi(\Omega))$ generated by this family of seminorms is called β-uniform topology. We denoted by $C_b(\Phi(\Omega))$ the closure of the algebra $C_\beta(\Phi(\Omega))$ in β-uniform topology and denote by $\mathcal{A}_\beta(\Omega)$ the closure of $\mathcal{A}_\infty(\Omega)$ with respect to the same topology [2–5]. So, we have $\Phi(\Omega) \subset M_{\mathcal{A}_\infty} \subset M_{C_b(\Phi(\Omega))} = b(\Phi(\Omega))$, where $b(\Phi(\Omega))$ is the Stone–Chekh compactification of the complete regular space $\Phi(\Omega)$.

Theorem 1. Let Ω be a topological space. Then the space of all β-continuous linear functionals of the symmetric β-uniformly closed algebra $\mathcal{A}_\beta(\Omega)$ with unit is isomorphic to the space $\mathcal{M}(\Phi(\Omega))$ of all finite complex regular measures on the complete regular space $\Phi(\Omega)$.

Proof. The space of maximal ideals $M_{\mathcal{A}_\infty}$ of the algebra $\mathcal{A}_\infty(\Omega)$ can be represented by $M_{\mathcal{A}_\infty} = \Phi(\Omega) \cup F$, where $F \cap \Phi(\Omega) = \emptyset$, F is a compact set and the border of $\Phi(\Omega)$ in $M_{\mathcal{A}_\infty}$. Since $\Phi(\Omega)$ is dense in $M_{\mathcal{A}_\infty}$, any function from the algebra $\mathcal{A}_\infty(\Omega)$ has unique norm-preserving extension to a function from the algebra $C(M_{\mathcal{A}_\infty})$. If φ is from $\mathcal{A}_\beta(\Omega)^*$, then φ is in $C(M_{\mathcal{A}_\infty})^*$. By Rise theorem, there exists a finite regular Borel measure μ on $M_{\mathcal{A}_\infty}$, representing the functional φ, i.e. $\varphi(f) = \int_{M_{\mathcal{A}_\infty}} \hat{f}\, d\mu$, where $\hat{f}$ is a Gelfand transformation of the function f in $\mathcal{A}_\infty(\Omega)$. Since $M_{\mathcal{A}_\infty} = \Phi(\Omega) \cup F$ the measure μ can be written by $\mu = \mu_{\Phi(\Omega)} + \mu_F$, where $\mu_{\Phi(\Omega)}$ and μ_F are the restrictions of the measure μ on $\Phi(\Omega)$ and F, respectively and $\mu_F = 0$. Let $\{e_j\}_{j\in J}$ be a bounded approximative unit in $\mathcal{B}_0(\Phi(\Omega)) \cap C(M_{\mathcal{A}_\infty})$. Then the sequence $f_j = 1 - e_j$ converges in β-uniform topology to the null function in the algebra $\mathcal{A}_\beta(\Omega)$. Hence, the linear functionals $\{f_j \circ \varphi\}_{j\in J}$, where $(f_j \circ \varphi)(f) = \varphi(f_j f)$ converges to the null functional. Thus, we have

$$0 = \lim_J (f_j \circ \varphi)(f) = \lim_J \Big(\int_{\Phi(\Omega)} \widehat{f_j f}\, d\mu + \int_F \widehat{f_j f}\, d\mu \Big) = \int_F \hat{f}\, d\mu.$$

Hence, for each function $f \in \mathcal{A}_\beta(\Omega)$ we have $\varphi(f) = \int_{\Phi(\Omega)} f\, d\mu$. □

Corollary 1. For any symmetric β-uniformly closed algebra $\mathcal{A}_\beta(\Omega)$ with unit there exists a linear multiplicative functional, which is discontinuous in β-uniform topology.

Proof. According to Theorem 1 for $\varphi \in M_{\mathcal{A}_\infty} \setminus \Phi(\Omega)$ the corresponding Dirac functional δ_φ, where $\delta_\varphi(f) = \hat{f}(\varphi)$ is a discontinuous linear multiplicative functional of the algebra $\mathcal{A}_\beta(\Omega)$.

We will state the following useful definition. Let $\mathcal{A}_\infty(\Omega)$ be a closed with respect to the sup-norm symmetric algebra with unit defined on a topological space Ω. Then the space of the maximal ideals $M_{\mathcal{A}_\infty}$ of $\mathcal{A}_\infty(\Omega)$ is said to be the $\mathcal{A}_\infty$-closure of the topological space Ω.

Proposition. Let $\mathcal{A}_\infty(\Omega)$ be a symmetric uniform algebra with unit defined on a topological space Ω. Then the topological space Ω is $\mathcal{A}_\infty$-dense in $M_{\mathcal{A}_\infty}$.

Proof. As it was stated above, $\mathcal{A}_\infty$-closure of a topological space Ω is the closure of the completely regular space $\Phi(\Omega)$ in $M_{\mathcal{A}_\infty}$, and since $\mathcal{A}_\infty(\Omega)$ is a uniform algebra, the sets Ω and $\Phi(\Omega)$ are topologically isomorphic. On the other hand, since the closure $\Phi(\Omega)$ coincides with the space of maximal ideals $M_{\mathcal{A}_\infty}$, we conclude that Ω is $\mathcal{A}_\infty$-dense in $M_{\mathcal{A}_\infty}$. □

Let G be a locally compact Abelian group, $\hat{G}$ is be its group of characters and let Γ be a nonempty subset in $\hat{G}$. We denote by $\mathcal{A}_\infty(\Gamma;G)$ a closed with respect to sup-norm symmetric subalgebra $C_b(G)$, with unit, which is generated by the family $\{1,\chi,\bar{\chi}\}_{\chi\in\Gamma}$. As was mentioned above $\mathcal{A}_\infty(\Gamma;G) = C\left(M_{\mathcal{A}_\infty(\Gamma;G)}\right)$, where $M_{\mathcal{A}_\infty(\Gamma;G)}$ is the space of maximal ideals of the algebra $\mathcal{A}_\infty(\Gamma;G)$. The arising "contraction" $\Phi_\Gamma : G \to M_{\mathcal{A}_\infty(\Gamma;G)}$, is defined by the formula $\Phi_\Gamma(x) = \varphi$, where $f(x) = \hat{f}(\varphi)$ for all $f \in \mathcal{A}_\infty(\Gamma;G)$ and $\hat{f}$ is the Gelfand transformation of the function f. We have $\Phi_\Gamma(G) \subset M_{\mathcal{A}_\infty(\Gamma,G)}$ and $\Phi_\Gamma(G)$ of the compact group $M_{\mathcal{A}_\infty(\Gamma;G)}$, which will be denoted by $b_\Gamma G$. Such $b_\Gamma G$ groups are called Bohr's compactifications of the group G. The group $b_{\hat{G}}G$ will be denoted by bG.

Thus, we have $\mathcal{A}_\infty(\Gamma;G) = \left\{f\circ\Phi_\Gamma \in C_b(G) : \hat{f} \in C\left(M_{\mathcal{A}_\infty(\Gamma,G)}\right)\right\}$. We note, that if the set Γ separates the points of G, then the algebra $\mathcal{A}_\infty(\Gamma;G)$ is a symmetric uniform algebra on a group G. In particular, if $\Gamma = \hat{G}$, then Γ separates the points of G and therefore, the mapping Φ_Γ is a topological homomorphism between the group G and $\Phi_\Gamma(G)$. Besides $\mathcal{A}_\infty(\hat{G};G) = C_{AP}(G)$, where $C_{AP}(G)$ is the uniform algebra of all almost periodical complex-valued continuous functions on a group G [6].

Theorem 2. The space $\mathcal{A}_\beta(\Gamma;G)^*$ of all β-continuous linear functionals on a symmetric β-uniformly closed algebras $\mathcal{A}_\beta(\Gamma;G)$ with unit is isomorph to the space $\mathcal{M}(\Phi_\Gamma(G))$ of all finite complex regular measures on a completely regular space $\Phi_\Gamma(G)$. If Γ separates the points of G, then the space $\mathcal{A}_\beta(\Gamma;G)^*$ is isomorphic to $\mathcal{M}(G)$.

Proof. The first part of the statement follows from Theorem 1. The second part is also immediate, because if Γ separates the points of G, then $\mathcal{A}_\beta(\Gamma;G)$ is a β-uniform algebra on G and therefore, the mapping Φ_Γ is a topological homomorphism between the groups G and $\Phi_\Gamma(G)$. Thus the group G is topologically imbedded into the compact group $b_\Gamma G = M_{\mathcal{A}_\infty(\Gamma,G)}$, where $b_\Gamma G$ is a Borov's compact of G, generated by Γ. Using Theorem 1, we get $\mathcal{A}_\beta(\Gamma;G)^* = \mathcal{M}(G)$. □

In the case $\Gamma = \hat{G}$ we obtain

Corollary 2. The space of all β-continuous linear functionals on a β-uniform algebra $C_{AP}(G)_\beta$, where G is a locally compact Abelian group, is isomorphic to the space $\mathcal{M}(G)$ of all finite complex regular measures on G.

From the Proposition we obtain the following corollary.

Corollary 3. Let $\mathcal{A}_\infty(\Gamma;G)$ be a symmetric uniform algebra with unit on a locally compact Abelian group G. Then the group G is $\mathcal{A}_\infty(\Gamma,G)$-dense in $b_\Gamma G$.

We denote by $M_{\mathcal{A}_\beta(\Gamma,G)}$ the set of all β-continuous linear multiplicative functionals on $\mathcal{A}_\beta(\Gamma,G)$. In the case $\Gamma=\hat{G}$ we denote by $M_{C_{AP}(G)_\beta}$ the set of all β-continuous linear multiplicative functionals on the β-uniform algebra $C_{AP}(G)_\beta$. From Theorem 2 it follows that $M_{C_{AP}(G)_\beta}=G$.

Corollary 4. If G is a locally compact Abelian group, then there exist discontinuous with respect to the β-uniform topology a linear multiplicative functional defined on the symmetric β-uniform algebra $C_{AP}(G)_\beta$.

Remark. Let $C^\tau_{AP}(G)$ be the space of all τ-almost periodical complex-valued continuous functions on a local compact Hausdorff space Ω, generating by the generalized shift operation τ^Ω (see [7, 8]). Since the space $C^\tau_{AP}(\Omega)$ separates the points of Ω, as mentioned above, $C^\tau_{AP}(G)_\beta$ is a β-uniform symmetric algebra on Ω. Thus, from Theorem 1 it follows that $C^\tau_{AP}(G)^*_\beta=\mathcal{M}(\Omega)$, since by Riss theorem we have $C^\tau_{AP}(G)^*=\mathcal{M}(b^\tau(\Omega))$, where $b^\tau(\Omega)$ is the Bohr's compactification of Ω generating by the generalized shift operation τ^Ω. In the case $\Omega=\mathbb{R}$, and $\tau^\mathbb{R}$ is a usual shift, then we have $C_{AP}(\mathbb{R})^*_\beta=\mathcal{M}(\mathbb{R})$.

Let us consider the following examples of highlight the above constructions of Bohr's compacts $b(\mathbb{R})$ on the real line $\mathbb{R}$.

Let $\{\lambda_j\}_{j=1}^n$ be an arbitrary finite family of nonzero real numbers such that the family $\{1,\lambda_1,\dots,\lambda_n\}$ is rational independent [9]. We denote by $\mathcal{A}_\infty\left(\{\lambda_j\}_{j=1}^n;\mathbb{R}\right)$ the symmetric closed in sup-norm subalgebra of $C_{AP}(\mathbb{R})$ having unit and generated by the family $\{e^{-i\lambda_1 t},\dots,e^{-i\lambda_n t},e^{i\lambda_1 t},\dots,e^{i\lambda_n t}\}$. The arising "contraction" mapping $\Phi_n:\mathbb{R}\to M_{\mathcal{A}_\infty\left(\{\lambda_j\}_{j=1}^n;\mathbb{R}\right)}$, which is given by the formula $\Phi_n(t)=\left(e^{i\lambda_1 t},\dots,e^{i\lambda_n t}\right)$ contracts the points $\left(e^{i\lambda_1 t},\dots,e^{i\lambda_n t}\right)$ and $\left(e^{i\lambda_1\left(t+\frac{2\pi k_1}{\lambda_1}\right)},\dots,e^{i\lambda_n\left(t+\frac{2\pi k_n}{\lambda_n}\right)}\right)$, where $t\in\mathbb{R}$, $k_j\in\mathbb{Z}$, $j=1,\dots,n$.

Since the space of maximal ideals of the algebra $\mathcal{A}_\infty\left(\{\lambda_j\}_{j=1}^n;\mathbb{R}\right)$ is n-dimentionals turns $T^n\subset\mathbb{C}^n$, then by Proposition we have $\Phi_n(\mathbb{R})$ is $\mathcal{A}_\infty$-dense in $M_{\mathcal{A}_\infty\left(\{\lambda_j\}_{j=1}^n;\mathbb{R}\right)}=T^n$. Thus, for all natural numbers n we have

$$\mathcal{A}_\infty\left(\{\lambda_j\}_{j=1}^n;\mathbb{R}\right)^*=\mathcal{M}(T^n),\ \text{and}\ \mathcal{A}_\beta\left(\{\lambda_j\}_{j=1}^n;\mathbb{R}\right)^*=\mathcal{M}(\Phi_n(\mathbb{R})).$$

Since $\mathcal{A}_\infty\left(\{\lambda_j\}_{j=1}^n;\mathbb{R}\right)=C(T^n)$, for all number n the uniform algebra $C(T^n)$ is a subalgebra of the uniform algebra $C_{AP}(\mathbb{R})$. From this it follows that the Bohr's compact $b(\mathbb{R})=M_{C_{AP}(\mathbb{R})}$ is compact in an infinite-measurable space and exactly, $b(\mathbb{R})\subset\mathbb{C}^{caard(\mathbb{R})}$, where $caard(\mathbb{R})$ is the continuum.

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