PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2015, № 3, p. 37–41

Informatics

INTERVAL NON-TOTAL COLORABLE GRAPHS

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A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. An interval total t-coloring of a graph G is a total coloring of G with colors $1,2,\ldots,t$ such that all colors are used, and the edges incident to each vertex v together with v are colored by $d_G(v)+1$ consecutive colors, where $d_G(v)$ is the degree of a vertex v in G. In this paper we describe some methods for constructing of graphs that have no interval total coloring.

MSC2010: 05C15.

***Keywords*:** total coloring, interval total coloring, interval coloring.

Introduction. All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G respectively. The degree of a vertex v in G is denoted by $d_G(v)(ord(v))$, the maximum degree of vertices in G by $\Delta(G)$ and the total chromatic number of G by $\chi''(G)$. A proper edge-coloring of a graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. If α is a proper edge-coloring of G and $v\in V(G)$, then $S(v,\alpha)$ denotes the set of colors appearing on edges incident to v. A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. For a total coloring α of a graph G and for any $v\in V(G)$, define the set $S[v,\alpha]$(spectrum of a vertex v) as follows:

$$S[v,\alpha]=\{\alpha(v)\}US(v,\alpha).$$

An interval total t-coloring [1] of a graph G is a total coloring α of G with colors $1,2,\ldots,t$ such that all colors are used and for any $v\in V(G)$, $S[v,\alpha]$ is an interval of integers. A graph G is interval total colorable, if it has an interval total t-coloring for some positive integer t. The set of all interval total colorable graphs is denoted by $\mathfrak{T}$. Terms and concepts that we do not define can be found in [2–4].

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The concept of interval total coloring was introduced by [1]. In [1, 5] the author proved, that if $r+s+2-gcd(r,s) \leq t \leq r+s+1$, then the complete bipartite graph $K_{r,s}$ has an interval total t-coloring, where $gcd(r,s)$ is the greatest common divisor of r and s. In [5], Petrosyan also investigated interval total colorings of complete graphs and n- dimensional cubes. Recently, Petrosyan and Khachatryan [6] have proved that the n-dimensional cube $Q_n (n \geq 3)$ has an interval total t-coloring, if and only if $n+1 \leq t \leq (n+1)(n+2)/2$. In [7–9] Petrosyan and Shashikyan investigated interval total colorings of bipartite graphs. In particular, they proved that all regular bipartite graphs, subcubic bipartite graphs, $(2,b)$ -biregular bipartite graphs and some classes of bipartite graphs with maximum degree 4 have interval total colorings. They also showed that there are bipartite graphs that have no interval total coloring. The smallest known bipartite graph with 26 vertices and maximum degree 18 that is not interval total colorable was obtained by Shashikyan [10]. In the present paper we describe some methods for constructing of graphs that have no interval total coloring.

Main Results. In this section three different methods for constructing of interval non-total colorable graphs will be presented.

1. Interval Non-Total Colorable Graphs Based on Shortest Paths.

Let us define the graph $P(n,k) (n,k \in N)$ as follows: $V(P(n,k)) = U \cup V$, where $U = \{a\} \cup \{c_{(i,j)} | 1 \leq i < j \leq k\}, V = \{b_j^{(i)} | 1 \leq i \leq k, 1 \leq j \leq n\}$,

$$E(P(n,k)) = \{(a, b_j^{(i)}) | 1 \leq i \leq k, 1 \leq j \leq n\} \cup$$
$$\{(c_{(i,j)}, b_i^{(i)}), (c_{(i,j)}, b_i^{(j)}) | 1 \leq i < j \leq k, 1 \leq l \leq n\}.$$

Clearly, $P(n,k)$ is a connected bipartite graph with $|V(P(n,k))| = = n \cdot k + \binom{k}{2} + 1,\ d(a) = \delta(P(n,k)) = n \cdot k, d(b_j^{(i)}) = k, d(c_{(i,j)}) = 2n,\ 1 \leq i \leq k,$ $1 \leq j \leq n$.

Theorem 1. If $n+k < \dfrac{n \cdot (k-1)}{2}$ $(n,k \in N)$, then $P(n,k) \notin \mathfrak{T}$.

Proof. Suppose, to the contrary, that the graph $P(n,k)$ has an interval total t-coloring α for some positive integer $t \geq n \cdot k + 1$. Suppose that $s = \min S(a, \alpha)$, $\alpha((a, b_{j_0}^{(i_0)})) = s$ and $\alpha((a, b_{j_1}^{(i_1)})) = \max S(a, \alpha) \geq s + n \cdot k - 1$. We consider two cases:

Case 1. $i_0 = i_1$.

In this case by the definition of $P(n,k)$, there is either $c_{(i_0,j)} \in V(P(n,k))$, when $i_0 < j$ or $c_{(j,i_0)} \in V(P(n,k))$, when $i_0 > j$, such that $(b_{j_0}^{(i_0)}, c_{(i_0,j)}), (b_{j_1}^{(i_0)}, c_{(i_0,j)}) \in E(P(n,k))$ or $(b_{i_0}^{(j_0)}, c_{(j,i_0)}), (b_{j_1}^{(i_0)}, c_{(j,i_0)}) \in E(P(n,k))$. Without loss of generality we may assume that there is $c_{(i_0,j)}) \in V(P(n,k)), i_0 < j$, and $(b_{j_0}^{(i_0)}, c_{(i_0,j)}, (b_{j_1}^{(i_0)}, c_{(i_0,j)} \in E(P(n,k))$. Since $d(b_{j_0}^{(i_0)}) = k,\ d(c_{(i_0,j)}) = 2n$, we have that $\alpha\left(\left(b_{j_0}^{(i_0)}, c_{(i_0,j)}\right)\right) \leq d+k$, so, $\alpha\left(\left(b_{j_1}^{(i_0)}, c_{(i_0,j)}\right)\right) \leq d+k+2n$. Clearly, $\alpha((a, b_{j_1}^{(i_1)})) \leq s+k+2n+k = s+2k+2n$. On the other hand, $\alpha\left(\left(a, b_{j_1}^{(i_1)}\right)\right) \geq s+n \cdot k - 1$, therefore, $s+n \cdot k-1 \leq s+2k+2n$, which means that $\dfrac{n \cdot k - 1}{2} \leq n+k$, a contradiction.

Case 2. $i_0 \neq i_1$.

In this case by the definition of $P(n,k)$, there is $c_{(i_0,i_1)} \in V(P(n,k))$ such that $(b_{j_0}^{(i_1)}, c_{(i_0,i_1)}), (b_{j_1}^{(i_1)}, c_{(i_0,i_1)}) \in E(P(n,k))$. Since $d(b_{j_0}^{(i_1)}) = k, d(c_{(i_0,i_1)}) = 2n$, we have that $\alpha((b_{j_0}^{(i_0)}, c_{(i_0,i_1)})) \leq s+k$, so, $\alpha((b_{j_1}^{(i_1)}, c_{(i_0,i_1)})) \leq s+k+2n$.

Clearly, $\alpha((a,b_{j_1}^{(i_1)})) \leq s+k+2n+k = s+2k+2n$. On the other hand $\alpha((a,b_{j_1}^{(i_1)})) \geq s+n\cdot k-1$, therefore, $s+n\cdot k-1 \leq s+2k+2n$, which means that $\frac{n\cdot k-1}{2} \leq n+k$, a contradiction. □

In Fig. 1 we can see the graph $P(4,5)$ that has no interval total coloring by Theorem 1.

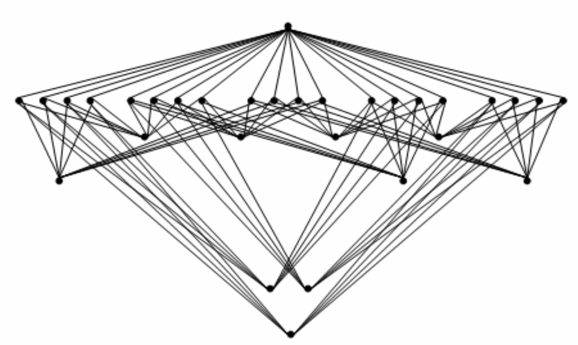

Fig. 1. Interval non-total colorable graph P(4, 5).

2. Interval Non-Total Colorable Graphs Based on Trees.

Let T be a tree, in which the distance between each two leaves is even and $F(T) = \{v | v \in V(T),\ d_T(v) = 1\}$. Let us define the graph $\hat{T}$ as follows:

$$V(\hat{T}) = V(T) \cup \{u\}, u \cup V(T),\ E(\hat{T}) = E(T) \cup \{(u,v) | v \in F(T)\}.$$

Clearly, $\hat{T}$ is a connected bipartite graph with $\delta(\hat{T}) = |F(T)|$.

Theorem 2. If T is a tree, in which the distance between each two leaves is even and $|F(T)| > 3 + \max_{P\in\mathbf{P}} \sum_{v\in V(P)} (d_T(v))$, then $\hat{T} \notin \mathfrak{T}$, where $\mathbf{P}$ is the set of paths connected each leaves in T.

Proof. Suppose, to the contrary, that the graph $\hat{T}$ has an interval total t-coloring α for some positive integer $t \geq |F(T)|+1$. Let $s = \min S(u,\alpha)$, $\alpha((u,v)) = s$ and $\alpha((u,v^{'})) = \max S(u,\alpha) \geq s+|F(T)|-1$. Since $\hat{T}-u$ is a tree, there is only one single path in $\hat{T}-u$, which connects the vertices v and $v^{'}$. Let $P_{vv'} = (v_1, e_1, v_2, ..., v_i, e_i, v_{i+1}, ..., v_k, e_k, v_{k+1})$, where $v_1 = v,\ v_{k+1} = v^{'}$. Clearly, if $1 \leq i \leq k$, then $\alpha((v_i, v_{i+1})) \leq s+1+\sum_{j=1}^{i}(d_T(v_j))$. So,

$$\alpha((v_k,v_{k+1})) = \alpha((v_k,v^{'})) \leq s+1+\sum_{j=1}^{k}(d_T(v_j)) \leq s+1+\max_{P\in\mathbf{P}} \sum_{v\in V(P)} (d_T(v)),$$

therefore, $s+|F(T)|-1 \leq \max S(u,\alpha) = \alpha((u,v^{'})) \leq s+2+\max_{P\in\mathbf{P}} \sum_{v\in V(P)} (d_T(v))$, which means that $|F(T)| \leq 3+\max_{P\in\mathbf{P}} \sum_{v\in V(P)} (d_T(v))$, a contradiction. □

In Fig. 2 we can see the tree T, for which the graph $\hat{T}$ has no interval total coloring by Theorem 2.

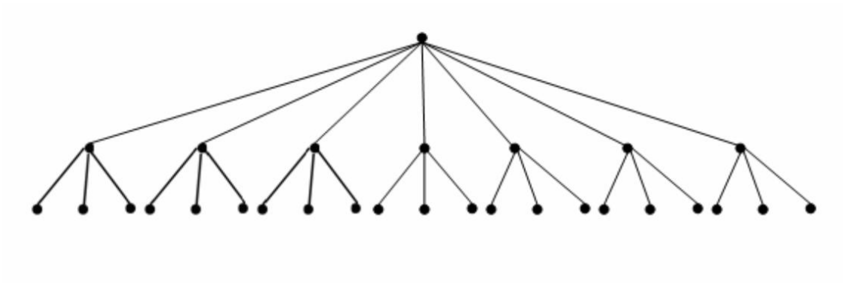

Fig. 2. Tree with even distance between each two leaves.

3. Interval Non-Total Colorable Graphs Based on Subdivisions.

Let G be a graph and $V(G) = \{v_1, v_2, ..., v_n\}$. Let us define graphs $S(G)$ and $\hat{G}$ as follows:

$$V(S(G)) = \{v_1, v_2, ..., v_n\} \cup \{w_{ij} : (v_i, v_j) \in E(G)\},$$

$$E(S(G)) = \{(v_i, w_{ij}), (v_j, w_{ij}) : (v_i, v_j) \in E(G)\},$$

$$V(\hat{G}) = V(S(G)) \cup \{u\}, u \notin V(S(G)),$$

$$E(\hat{G}) = E(S(G)) \cup (u, w_{ij}) | (v_i, v_j) \in E(G).$$

In other words, $S(G)$ is the graph obtained by subdividing every edge of G, and $\hat{G}$ is the graph obtained from $S(G)$ by connecting every inserted vertex to a new vertex u. Clearly, $S(G)$ and $\hat{G}$ are bipartite graphs.

Theorem 3. If G is a connected graph and $|E(G)| > 1+\max_{P\in\mathbf{P}} \sum_{v\in V(P)} (d_{\hat{G}}(v))$, where $\mathbf{P}$ is a set of all shortest paths in $S(G)$ connecting vertices w_{ij}, then $\hat{G} \notin \mathfrak{T}$.

Proof. Suppose, to the contrary, that the graph $\hat{G}$ has an interval total t-coloring α for some positive integer $t \geq |E(G)|$. Consider the vertex u. Let w and $w^{'}$ be two vertices adjacent to u such that $\alpha((u,w)) = \min S(u,\alpha) = s$ and $\alpha((u,w1)) = \max S(u,\alpha) \geq s+|E(G)|-1$. Since $\hat{G}-u = S(G)$ and $\hat{G}-u$ is connected, there is a shortest path $P(w,w^{'})$ in $\hat{G}-u$ joining w and $w^{'}$, where

$$P(w,w^{'}) = (x_1, e_1, x_2, ..., x_i, e_i, x_{i+1}, ..., x_k, e_k, x_{k+1}),\ x_1 = w,\ x_{k+1} = w^{'}.$$

Note that

$$\alpha((x_i, x_{i+1})) \leq s + \sum_{j=1}^{i} (d_{\hat{G}}(x_j)$$

for $1 \leq i \leq k$, and

$$\alpha((x_{k+1},u)) = \alpha((w^{'},u)) \leq s + \sum_{j=1}^{k+1}(d_{\hat{G}}(x_j)).$$

Hence,

$$s+|E(G)|-1 \leq \max S(u,\alpha) = \alpha((u,w^{'})) \leq s+\sum_{j=1}^{k+1}(d_{\hat{G}}(x_j)) \leq s+\max_{P\in\mathbf{P}} \sum_{v\in V(P)}(d_{\hat{G}}(v))$$

and thus $|E(G)| \leq 1+\max_{P\in\mathbf{P}} \sum_{v\in V(P)}(d_{\hat{G}}(v))$, which is contradiction. □

Author would like to express his gratitude to P. Petrosyan for useful discussions over the subject.

Received 23.04.2015

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