PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2016, № 1, p. 7–11

Mathematics

A DIFFERENTIATION AND DIVIDED DIFFERENCE FORMULA FOR RATIONAL FUNCTIONS

G. Ye. MKRTCHYAN *, Ye. S. MKRTCHYAN **

Chair of Numerical Analysis and Mathematical Modeling YSU, Armenia

In this paper a new differentiation and divided difference formula for rational functions is proved. The main result is a connection between divided differences of two rational functions with the same numerator, where the knots of one divided difference coincide with the zeros of the denominator of another rational function.

MSC2010: Primary 41A20; Secondary 65D05.

***Keywords*:** rational function, divided difference.

Introduction. Denote the space of all polynomials of degree $\leq n$ by π_n:

$$\pi_n = \left\{ \sum_{i\leq n} a_i x^i \right\}.$$

We have that

$$\dim \pi_n = n+1.$$

Consider a set of $n+1$ distinct knots (points)

$$\{a_0, a_1, \ldots, a_n\} \subset \mathbb{R}.$$

The problem of finding a polynomial $p \in \pi_n$, which satisfies the conditions

$$p(a_i) = c_i, \quad i = 0, \ldots, n, \tag{1}$$

is called interpolation problem. It is well-known that for any data $\{c_i,\ i = 0, \ldots, n\}$ there exists a unique polynomial $p \in \pi_n$ satisfying the conditions (1) (see [1,2]).

The polynomial from π_n satisfying the conditions

$$p(a_i) = f(a_i), \quad i = 0, \ldots, n,$$

is called interpolation polynomial of f. Let us denote it by

$$p_f := p_{f,n} := p_{f,n,a_0,\ldots,a_n}.$$

* E-mail: geworg306@gmail.com ** E-mail: ervand48@mail.ru

We have the following Lagrange formula for the interpolation polynomial:

$$p_f = \sum_{i=0}^{n} f(a_i) p_i^*, \tag{2}$$

where $p_i^*(x) = \prod_{j\in\{0,\dots,n\}\setminus\{i\}} \frac{x-a_j}{a_i-a_j}$.

The divided difference of f with respect to the set of knots $\{a_0,\dots,a_n\}$ denoted by $[a_0,\dots,a_n]f$ is defined as the coefficient of x^n in p_f (see [1], Section 1.3):

$$[a_0,\dots,a_n]f = \textit{the leading coefficient of } p_f.$$

Therefore, we get from (2) that

$$[a_0,\dots,a_n]f = \sum_{i=0}^{n} \frac{f(a_i)}{\prod_{j\in\{0,\dots,n\}\setminus\{i\}} (a_i-a_j)}.$$

Note that usually the divided differences are defined also through the familiar recurrence relation:

$$[a_0,\dots,a_n]f = \frac{[a_1,\dots,a_n]f - [a_0,\dots,a_{n-1}]f}{a_n-a_0} \text{ and } [a]f = f(a).$$

The following well-known divided difference formula for the function $\frac{1}{x-b}$ will be used in the sequel

$$[a_0,\dots,a_n]\frac{1}{x-b} = \frac{(-1)^n}{(a_0-b)\cdots(a_n-b)}. \tag{3}$$

Note that in view of the recurrence relation, it can be readily proved by induction on n.

The divided difference with repeated (multiple) knots $\{x_0,\dots,x_n\}$ is defined as a limit of the divided differences with distinct knots:

$$[x_0,\dots,x_n]f = \lim_{m\to\infty}[a_0^{(m)},\dots,a_n^{(m)}]f.$$

Here the knots of the divided difference in the right hand side are distinct for each m and $a_i^{(m)} \to x_i$ when $m\to\infty$. It is also assumed that the function f is smooth enough, say $f \in C^{(n)}$.

The following relation between the divided difference and higher order derivative is important:

$$f^{(n)}(x) = n![x,\dots,x]f, \tag{4}$$

where x in the divided difference is repeated $n+1$ times.

By applying this relation to divided difference formulas one gets readily the respective differentiation formulas. For example, in this way, by setting $a_0 = \dots = a_n = x$ in (3), we get the following differentiation formula:

$$\left[\frac{1}{x-b}\right]^{(n)} = \frac{n!(-1)^n}{(x-b)^n}.$$

The Main Result.

The Divided Difference Formula.

Proposition 1. Let p be a polynomial from π_{m+n+1} with the leading coefficient γ. Suppose also that $q_1(x) = (x-b_0)\cdots(x-b_m)$ and $q_2(x) = (x-a_0)\cdots(x-a_n)$, where a_i are different from b_j.

Then we have

$$[a_0,\ldots,a_n]\frac{p}{q_1} = \gamma - [b_0,\ldots,b_m]\frac{p}{q_2}.$$

In particular, we have

$$[a_0,\ldots,a_n]\frac{p}{q_1} = -[b_0,\ldots,b_m]\frac{p}{q_2}, \text{ if } p \in \pi_{m+n}.$$

Proof. Assume that $p \in \pi_k$ and divide it by q_1 to get

$$p = sq_1 + r, \tag{5}$$

where $s \in \pi_{k-m-1}$ and $r \in \pi_m$. Note that

$$p(b_i) = r(b_i), \quad i = 0,\ldots,m. \tag{6}$$

Now we have

$$[a_0,\ldots,a_n]\frac{p}{q_1} = [a_0,\ldots,a_n]s + [a_0,\ldots,a_n]\frac{r}{q_1}.$$

Notice that if p is a polynomial from π_{m+n+1}, then $s \in \pi_n$ and the leading coefficients of p and s coincide. Thus $[a_0,\ldots,a_n]s = \gamma$.

Next, the rational function $\frac{r}{q_1}$ is a proper quotient.

Let us present it in the form of sum of simple quotients:

$$\frac{r}{q_1} = \sum_{i=0}^{m} \frac{A_i}{x-b_i},$$

where

$$A_i = \frac{r(b_i)}{\prod\limits_{j\in\{0,\ldots,m\}\setminus\{i\}} (b_i - b_j)}. \tag{7}$$

Note that the simple quotient representation actually coincides with the Lagrange formula (2).

Now, by using the formula (3), we get

$$[a_0,\ldots,a_n]\frac{r}{q_1} = \sum_{i=0}^{m} A_i \frac{(-1)^n}{(a_0-b_i)\cdots(a_n-b_i)} = -\sum_{i=0}^{m} \frac{A_i}{(b_i-a_0)\cdots(b_i-a_n)}.$$

Next, in view of formulas (6) and (7), we obtain

$$[a_0,\ldots,a_n]\frac{r}{q_1} = -\sum_{i=0}^{m} \frac{r(b_i)}{\prod\limits_{j\in\{0,\ldots,m\}\setminus\{i\}} (b_i-b_j)} \cdot \frac{1}{(b_i-a_0)\cdots(b_i-a_n)} =$$

$$= -\sum_{i=0}^{m} \frac{p(b_i)}{q_2(b_i)} \cdot \frac{1}{\prod\limits_{j\in\{0,\ldots,m\}\setminus\{i\}} (b_i-b_j)} = -[b_0,\ldots,b_m]\frac{p}{q_2}. \qquad \square$$

Notice that actually we proved the following more general result:

Proposition 2. Let p be a polynomial from π_k. Suppose also that $q_1(x) = (x-b_0)\cdots(x-b_m)$ and $q_2(x) = (x-a_0)\cdots(x-a_n)$, where a_i are different from b_j. Then we have

$$[a_0,\ldots,a_n]\frac{p}{q_1} = [a_0,\ldots,a_n]s - [b_0,\ldots,b_m]\frac{p}{q_2},$$

where the polynomial $s \in \pi_{k-m-1}$ is the quotient of p and q_1 defined by the equality (5).

The Differentiation Formula. Let us set $a_0 = \cdots = a_n = x$ in Proposition 2. Then, in view of the formula (4), we readily get the following differentiation result for rational functions:

Proposition 3. Let p be a polynomial from π_k. Suppose also that $q(x) = (x-b_0)\cdots(x-b_m)$.

Then we have

$$\left(\frac{p}{q}\right)^{(n)}(x) = s^{(n)}(x) - n![b_0,\ldots,b_m]\frac{p(\cdot)}{(\cdot - x)^{n+1}},$$

where the divided difference is with respect to the variable $\cdot$, and the polynomial $s \in \pi_{k-m-1}$ is the quotient of p and q_1 defined by the equality (5).

Some Special Cases.

The Case $m = 0$.

Consider the following special case of Proposition 1: $p(x) \equiv 1$ and $m = 0$, i.e. $q_1(x) = (x-b)$. Then we have $\gamma = 0$ and, therefore,

$$[a_0,\ldots,a_n]\frac{1}{x-b} = -\frac{1}{q_2(b)} = -\frac{1}{(b-a_0)\cdots(b-a_n)} = \frac{(-1)^n}{(a_0-b)\cdots(a_n-b)}.$$

Thus, we get the formula (3). Next, let p be any polynomial from π_n. Now, again we have $\gamma = 0$ and

$$[a_0,\ldots,a_n]\frac{p(x)}{x-b} = -\frac{p(b)}{q_2(b)} = \frac{(-1)^n p(b)}{(a_0-b)\cdots(a_n-b)}.$$

In the case of a polynomial p from π_{n+1} with the leading coefficient γ we get

$$[a_0,\ldots,a_n]\frac{p(x)}{x-b} = \gamma - \frac{p(b)}{q_2(b)} = \gamma + \frac{(-1)^n p(b)}{(a_0-b)\cdots(a_n-b)}.$$

From here, by taking $a_0 = \cdots = a_n = x$, we get the following differentiation formulas:

$$\left[\frac{p(x)}{x-b}\right]^{(n)} = \frac{(-1)^n n! p(b)}{[(x-b)]^{n+1}} \text{ for any } p \in \pi_n,$$

$$\left[\frac{p(x)}{x-b}\right]^{(n)} = \gamma n! + \frac{(-1)^n n! p(b)}{[(x-b)]^{n+1}} \text{ for any } p \in \pi_{n+1}.$$

Let us consider also

The Case $m = 1$.

Now consider the special case $m = 1$ of Proposition 1.

Let p be any polynomial from π_n. Then we have $\gamma = 0$ and

$$[a_0, \ldots, a_n] \frac{p(x)}{(x-b_0)(x-b_1)} =$$

$$= \frac{(-1)^n}{b_1 - b_0} \left[\frac{p(b_1)}{(a_0 - b_1)\cdots(a_n - b_1)} - \frac{p(b_0)}{(a_0 - b_0)\cdots(a_n - b_0)} \right].$$

In the case of polynomial from π_{n+1} with the leading coefficient γ we get

$$[a_0, \ldots, a_n] \frac{p(x)}{(x-b_0)(x-b_1)} =$$

$$= \gamma + \frac{(-1)^n}{b_1 - b_0} \left[\frac{p(b_1)}{(a_0 - b_1)\cdots(a_n - b_1)} - \frac{p(b_0)}{(a_0 - b_0)\cdots(a_n - b_0)} \right].$$

From here, by taking $a_0 = \cdots = a_n = x$, we get the following differentiation formulas:

$$\left[\frac{p(x)}{(x-b_0)(x-b_1)} \right]^{(n)} = \frac{(-1)^n n!}{b_1 - b_0} \left[\frac{p(b_1)}{[(x-b_1)]^{n+1}} - \frac{p(b_0)}{[(x-b_0)]^{n+1}} \right]$$

for any $p \in \pi_n$,

$$\left[\frac{p(x)}{(x-b_0)(x-b_1)} \right]^{(n)} = \gamma n! + \frac{(-1)^n n!}{b_1 - b_0} \left[\frac{p(b_1)}{[(x-b_1)]^{n+1}} - \frac{p(b_0)}{[(x-b_0)]^{n+1}} \right]$$

for any $p \in \pi_{n+1}$.

Received 25.12.2015

REFERENCES

1. **Bojanov B.D., Hakopian H.A., Sahakian A.A.** Spline Functions and Multivariate Interpolations. Dordrecht–Boston–London: Kluwer Academic Publishers, 1993.
2. **de Boor C.** A Practical Guide to Splines. Series: Applied Mathematical Sciences. Springer, 1978.