Let the sequence \( \{a_k\}_{k=1}^{\infty}, a_k \searrow 0 \) with \( \{a_k\}_{k=1}^{\infty} \notin l_2 \), and Walsh system \( \{W_k(x)\}_{k=0}^{\infty} \) be given. Then for any \( \epsilon > 0 \) there exists a measurable set \( E \subset [0,1] \) with measure \( |E| > 1 - \epsilon \) and numbers \( \delta_k = \pm 1, 0 \) such that for any \( p \in [1,2] \) and each function \( f(x) \in L^p(E) \) there exists a rearrangement \( k \to \sigma(k) \) such that the series \( \sum_{k=1}^{\infty} \delta_{\sigma(k)} a_{\sigma(k)} W_{\sigma(k)}(x) \) converges to \( f(x) \) in the norm of \( L^p(E) \).

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**Introduction.** A series

\[
\sum_{k=1}^{\infty} f_k(x), \{f_n(x)\}_{n=1}^{\infty} \in L^p[0,1], p \in [1,\infty),
\]

(A)

is said to be quasi universal in \( L^p[0,1] \) with respect to rearrangements, if for any \( \epsilon > 0 \) there exists a measurable set \( E \subset [0,1] \) with measure \( |E| > 1 - \epsilon \) such that for any \( p \geq 1 \) and each function \( f(x) \in L^p(E) \) there exists a rearrangement \( k \to \sigma(k) \) such that the series \( \sum_{k=1}^{\infty} f_{\sigma(k)}(x) \) converges to \( f(x) \) in the norm of \( L^p(E) \).

The question of existence of various types of universal series in the sense of almost everywhere or in measure convergence have been considered in [1]-[8]. The first trigonometric series universal in the usual sense in the class of all measurable functions for convergence almost everywhere has been constructed by Men’shov [1]. This result was extended by Talalyan [2] to arbitrary orthonormal complete systems. Also was established, that if \( \{\phi_n(x)\}_{n=1}^{\infty}, x \in [0,1], \) is an arbitrary complete orthonormal system, then there exists a series \( \sum_{k=1}^{\infty} a_k \phi_k(x), \) which is universal with respect

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to the convergence in measure of the partial series in the class of all measurable functions on \([0, 1]\).

In [3] the following theorems were proved:

**Theorem 1.** For any complete orthonormal system \(\{\varphi_n(x)\}_{n=1}^\infty\) there exists a series \(\sum_{k=1}^\infty b_k\varphi_k(x)\) with \(\sum_{k=1}^\infty |b_k|^r < \infty\) for any \(r > 2\), which is quasi universal in all spaces \(L^p[0, 1]\) for \(p \in [1, 2]\) simultaneously with respect to both rearrangements and partial series.

**Theorem 2.** Let the sequence \(\{a_k\}_{k=1}^\infty, a_k \searrow 0\) with \(\{a_k\}_{k=1}^\infty \notin l_2\), and the Walsh system \(\{W_k(x)\}_{k=0}^\infty\) be given. Then there exist numbers \(\delta_k = \pm 1\) such that the series \(\sum_{k=1}^\infty \delta_k a_k W_k(x)\) is universal with respect to a.e. convergence of the partial series in the class of all measurable a.e. finite functions on \([0, 1]\).

In this paper we prove the following theorem.

**Theorem 3.** Let the sequence \(\{a_k\}_{k=1}^\infty, a_k \searrow 0\) with \(\{a_k\}_{k=1}^\infty \notin l_2\), and the Walsh system \(\{W_k(x)\}_{k=0}^\infty\) be given. Then for any \(\varepsilon > 0\) there exist a measurable set \(E \subset [0, 1]\) with measure \(|E| > 1 - \varepsilon\) and numbers \(\delta_k = \pm 1\) such that for any \(p \in [1, 2]\) and each function \(f(x) \in L^p(E)\) there exists a rearrangement \(k \to \sigma(k)\) such that the series \(\sum_{k=1}^\infty \delta_{\sigma(k)} a_{\sigma(k)} W_{\sigma(k)}(x)\) converges to \(f(x)\) in the norm of \(L^p(E)\).

First we show that for any \(p \geq 2\) \((p \geq 1)\) and for any orthonormal (bounded orthonormal) system \(\{\varphi_n(x)\}\) there is no series, that is universal either with respect to rearrangements or signs in \(L^p[0, 1]\) \((p > 2\) we assume that \(\varphi_n(x) \in L^p[0, 1], n = 1, 2, \ldots\)).

Indeed, if for some \(p_0 \geq 2\) \((p_0 \geq 1)\) and for some orthonormal (bounded orthonormal) system \(\{\varphi_n(x)\}\) there exists a series \((t^0)\), which is universal in \(L^{p_0}[0, 1]\) with respect to rearrangements, then according to Eq. (A) for \(|a_1| + 1)\varphi_1(x)\) there exists a rearrangement \(k \to \sigma(k)\) such that

\[
\lim_{m \to \infty} \left\| \sum_{k=1}^m a_{\sigma(k)} \varphi_{\sigma(k)}(x) - (|a_1| + 1)\varphi_1(x) \right\|_{L^{p_0}} = 0,
\]

or since \(\varphi_1(x) \in L^{p_0}[0, 1]\) \(\left(\frac{1}{q_0} + \frac{1}{p_0} = 1\right.\) for \(p_0 > 1\) and \(q_0 = \infty\) for \(p_0 = 1\)\), for the first case we have \(a_1 = 1 + |a_1|\), while for the second case

\[
1 + |a_1| = \begin{cases} a_1 & \text{for } n_1 = 1, \\ 0 & \text{for } n_1 > 1. \end{cases}
\]

This contradiction completes the Proof.

**Proof of Basic Lemmas.** We shall use the following lemma of the paper [5].

**Lemma 1.** Let the sequence \(\{a_k\}_{k=1}^\infty, a_k \searrow 0\) with \(\{a_k\}_{k=1}^\infty \notin l_2\), the numbers \(n_0 > 1\) \((n_0 \in \mathbb{N}\) \(\not= 0\), \(\varepsilon > 0, \delta > 0\) and the interval of the form \(\Delta = \Delta_m^{(k)} = \left[\frac{k-1}{2^m}, \frac{k}{2^m}\right), k \in [1, 2^m]\), be given. Then, there exist a measurable set
$E \subset \Delta$ and polynomials $H(x), Q(x)$ in Walsh system of the form
\[
H(x) = \sum_{k=0}^{N} b_k W_k(x), \quad Q(x) = \sum_{k=0}^{N} \delta_k a_k W_k(x),
\]
satisfying the conditions:

- a) $\delta_k = \pm 1, 0$;
- b) $|E| = |\Delta|(1 - \varepsilon)$;
- c) $H(x) = 0, \ x \in [0, 1] \setminus \Delta$;
- d) $|Q(x) - \gamma| < \delta, \ x \in E$;
- e) $\max_{N_0 \leq n < N} \left| \sum_{k=0}^{n} b_k W_k(x) \right| < C |\gamma| + \delta, \ x \in \Delta$ ($C$ is an absolute constant);
- f) $\int_0^1 |Q(x) - H(x)|^2 dx < \varepsilon \delta^2 |\Delta|$;
- g) $\max_{N_0 \leq n < N} \left| \sum_{k=0}^{n} b_k W_k(x) \right| < \delta, x \in [0, 1] \setminus \Delta$.

Using Lemma 1, we prove the following lemma, which is the basic tool in the Proof of Theorem 3.

**Lemma 2.** Let the sequence $\{a_k\}_{k=1}^{\infty}, a_k \searrow 0$ with $\{a_k\}_{k=1}^{\infty} \notin l_1$, be given. Then, for any numbers $N_0 \in \mathbb{N}, \ 0 < \delta, \ \varepsilon < 1, p \in [1, 2]$ and for each polynomial $f(x)$ in Walsh system with $|f| > 0$ one can find a measurable set $E \subset [0, 1]$ and a polynomial in Walsh system of the form
\[
Q(x) = \sum_{k=0}^{N} \delta_k a_k W_k(x) \text{ when } \delta_k = \pm 1
\]
satisfying the conditions:

1) $|E| = 1 - \varepsilon$;
2) $\left( \int_E |Q(x) - f(x)|^2 dx \right)^{1/2} < \varepsilon$;
3) $\max_{N_0 \leq n < N} \left( \int_E \left| \sum_{k=0}^{n} \delta_k a_k W_k(x) \right|^p dx \right)^{1/p} \leq \left( \int_E |f(x)|^p dx \right)^{1/p} + \varepsilon, \ \forall p \in [1, 2]$.

**Proof of Lemma 2.** Let
\[
f(x) = \sum_{v=1}^{\mu_0} \psi_v \chi_{\Delta_v}(x), \quad \{0, 1\} = \bigcup_{v=1}^{\mu_0} \Delta_v,
\]
where $\Delta_v, \ v = 1, 2, \ldots, \mu_0$, are disjoint diadic intervals ($\chi_\Delta$ is characteristic function of $\Delta$). We choose a number $\beta > 0$ such that
\[
\max \left\{ \beta, \beta \sum_{v=1}^{\mu_0} |\Delta_v|^{1/2} \right\} < \min \left\{ \frac{\delta}{4}; C \min |f|; \left( \int_0^1 |f|^p dx \right)^{1/p} \right\}.
\]
Consecutively applying Lemma 1, one can define sets $E_v \subset \Delta_v$ for $1 \leq v \leq \mu_0$ and polynomials

$$H_v(x) = \sum_{k=N_0}^{N_v-1} b_k W_k(x),$$  \hspace{1cm} (3)$$

$$Q_v(x) = \sum_{k=N_0}^{N_v-1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1,$$  \hspace{1cm} (4)$$

where $N_0 = n_0$, satisfying the conditions:

$$|E_v| = |\Delta_v|(1 - \varepsilon),$$  \hspace{1cm} (5)$$

$$H_v(x) = 0, \quad x \in [0, 1] \setminus \Delta_v,$$  \hspace{1cm} (6)$$

$$|H_v(x) - \gamma_v| < \beta, \quad x \in E_v,$$  \hspace{1cm} (7)$$

$$\max_{N_{v-1} \leq n < N_v} \left| \sum_{k=N_0}^{n} b_k W_k(x) \right| < C |\gamma_v| + \beta, \quad x \in \Delta_v \quad (C > 1),$$  \hspace{1cm} (8)$$

$C$ is an absolute constant,

$$\int_0^1 |Q_v(x) - H_v(x)|^2 dx < \varepsilon \beta^2 |\Delta_v|,$$  \hspace{1cm} (9)$$

$$\max_{N_{v-1} \leq n < N_v} \left| \sum_{k=N_0}^{n} b_k W_k(x) \right| < \beta, \quad x \in [0, 1] \setminus \Delta_v.$$  \hspace{1cm} (10)$$

We put

$$E = \bigcup_{v=1}^{\mu_0} E_v,$$  \hspace{1cm} (11)$$

$$Q(x) = \sum_{v=1}^{\mu_0} Q_v(x) = \sum_{v=1}^{\mu_0} \sum_{k=N_0}^{N_v-1} \delta_k a_k W_k(x) = \sum_{k=N_0}^{N} \delta_k a_k W_k(x),$$  \hspace{1cm} (12)$$

$$H(x) = \sum_{v=1}^{\mu_0} H_v(x) = \sum_{v=1}^{\mu_0} \sum_{k=N_0}^{N_v-1} a_k W_k(x) = \sum_{k=N_0}^{N} a_k W_k(x).$$  \hspace{1cm} (13)$$

From (5) and (11) it follows that $|E| = 1 - \varepsilon$. Using equations (3), (4), (12), (13) and (9), we have

$$\left( \int_0^1 |Q(x) - H(x)|^2 dx \right)^{1/2} \leq \sum_{v=1}^{\mu_0} \left( \int_0^1 |Q_v(x) - H_v(x)|^2 dx \right)^{1/2} \leq$$

$$\leq \beta \sum_{v=1}^{\mu_0} (\varepsilon |\Delta_v|)^{1/2} \leq \min \left\{ \delta \frac{\varepsilon}{4} \left( \int_0^1 |f|^p dx \right)^{1/p} \right\}.$$  \hspace{1cm} (14)$$

Then, according to (1), (2) and (6), (7), (9), we get
From this and from relations (12)–(14) we conclude that
\[
\left( \int_E |H(x) - f(x)|^2 dx \right)^{1/2} \leq \sum_{v=1}^{\mu_0} \left( \int_{E_v} |H_v(x) - \gamma_v|^2 dx \right)^{1/2} \leq \sum_{v=1}^{\mu_0} \left( \int_{E_v} |Q_v(x) - \gamma_v|^2 dx \right)^{1/2} < (15)
\]
\[
< \beta \sum_{v=1}^{\mu_0} (|\Delta_v|)^{1/2} + \beta \sum_{v=1}^{\mu_0} |\Delta_v|^{1/2} < \delta.
\]
From this and from relations (12)–(14) we conclude that
\[
\left( \int_E |Q(x) - f(x)|^2 dx \right)^{1/2} \leq \left( \int_0^1 |Q(x) - H(x)|^2 dx \right)^{1/2} + \left( \int_E |H(x) - f(x)|^2 dx \right)^{1/2} < \delta,
\]
which proves the condition 3) of the Lemma.

If \( n \in [N_0, N] \), then for some \( 1 \leq v \leq \mu_0 \) and \( n \in [N_{v-1}, N_v) \) from (14) we get
\[
\sum_{k=N_0}^n \delta_k a_k W_k(x) = \sum_{j=1}^{v-1} \sum_{k=N_{j-1}}^{N_j} \delta_k a_k W_k(x) + \sum_{k=N_{v-1}}^n \delta_k a_k W_k(x),
\]
From this and (4), (6), (8), (13), (14), for any \( p \in [1,2] \) we obtain
\[
\left( \int_E \left| \sum_{k=N_0}^n \delta_k a_k W_k(x) \right|^p dx \right)^{1/p} \leq \sum_{j=1}^{v-1} \left( \int_E |Q_j(x) - H_j(x)|^p dx \right)^{1/p} + \sum_{j=1}^{v-1} \left( \int_E |H_j(x)|^p dx \right)^{1/p} + \\
\left( \int_E \left| \sum_{k=N_{v-1}}^n (\delta_k a_k - a_k W_k(x)) \right|^p dx \right)^{1/p} + \left( \int_E \left| \sum_{k=N_{v-1}}^n a_k W_k(x) \right|^p dx \right)^{1/p} \leq \\
\beta \sum_{j=1}^{v-1} (|\Delta_v|)^{1/2} + 2 \left( \sum_{v=1}^{\mu_0} (|\gamma_v + \beta|)^p |\Delta_v| \right)^{1/p} + \left( \sum_{v=1}^{\mu_0} |\gamma_v| \right)^{1/p} + \beta \left( \sum_{v=1}^{\mu_0} |\Delta_v| \right)^{1/p} \leq \\
\leq 2 \left( \int_E |f(x)|^p dx \right)^{1/p} + \varepsilon. \quad \square
\]

**Proof of Theorem 3.** Let
\[
\left\{ f_k(x) \right\}_{k=1}^\infty
\]
be the system of all algebraic polynomials with rational coefficients and the sequence \( \{a_k\}_{k=1}^\infty \), \( a_k \to 0 \) with \( \{a_k\}_{k=1}^\infty \notin l_2 \), and \( \varepsilon \in (0,1) \) be given.
Applying Lemma 2 when \( f = f_n \) from (17), we determine a sequence of measurable sets \( \{E_n\} \) and a sequence of polynomials

\[
Q_n(x) = \sum_{k=m_{n-1}}^{m_n-1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1, 0, \quad m_0 = 2, \quad m_n > m_{n-1},
\]

where \( m_0 \) is the integer part of \( \log \frac{1}{2} \varepsilon \). From (19) and (23) we get \( |E| > 1 - \varepsilon \).

Let \( f(x) \in L^p(E) \), \( p \in [1, 2] \). Consider a function \( f_{n_1}(x) \) \( (n_1 > n_0) \) from (2) satisfying

\[
\| f(x) - f_{n_1}(x) \|_{L^p(E)} = \left( \int_E |f(x) - f_{n_1}(x)|^p \, dx \right)^{1/p} < 2^{-4(1+1)}.
\]

From this and (20)

\[
\| f(x) - [Q_{n_1}(x) + a_1 W_1(x)] \|_{L^p(E)} < 2^{-2n_1} + 2^{-8} + a_1.
\]

Assume that the members

\[
\left\{ \{ \delta_j = \pm 1, k_j \}_{j=m_{n-1}}^{m_n-1}, q_s \right\}_{s=1}^v
\]

and functions

\[
\left\{ \delta_k a_k W_k(x) \right\}_{k=m_{n-1}}^{m_{n+1}}, a_{q_1} W_{q_1}(x), \quad s = 1, 2, \ldots, v,
\]

of the series (22) or (23) are chosen to satisfy

\[
q_v = \min \left\{ \bigcup_{s=1}^v \left\{ k \right\}_{k=m_{n-1}}^{m_{n+1}} \cup \{ q_1, \ldots, q_{v-1} \} \right\}.
\]

\[
\left\| f(x) - \sum_{j=1}^v [Q_{q_j}(x) + a_{q_j} W_{q_j}(x)] \right\|_{L^p(E)} \leq 2^{-2n_v} + 2^{-4(v+1)} + a_{q_v},
\]

(24)
\[
\max_{m_{n_\nu-1} \leq m \leq m_{n_\nu}} \left\| \sum_{k=m_{n_\nu-1}}^{m} \delta \alpha_k W_k(x) \right\|_{L^p(E)} < 2^{-\nu} + a_{q_\nu}, \quad \forall p \in [1, 2]. \tag{25}
\]

We choose a natural number \(n_{\nu+1} > n_\nu\) and a function \(f_{n_{\nu+1}}(x)\) from (2) satisfying
\[
\left\| f_{n_{\nu+1}}(x) - f(x) + \sum_{j=1}^{\nu} [Q_{n_j}(x) + a_{q_j} W_{q_j}(x)] \right\|_{L^p(E)} < 2^{-4(\nu+1)}. \tag{26}
\]

From this and from (25) we obtain
\[
\left\| f_{n_{\nu+1}}(x) \right\|_{L^p(E)} \leq 2^{-2(\nu+1)} + a_{q_\nu}.
\]

Therefore, by (18), (21) and (23) we have
\[
\max_{m_{n_{\nu+1}-1} \leq N \leq m_{n_{\nu+1}}} \left\| \sum_{k=m_{n_{\nu+1}-1}}^{N} \delta k W_k(x) \right\|_{L^p(E)} < 2^{-(\nu+1)} + a_{q_\nu}, \tag{26}
\]

\[
\left\| Q_{n_{\nu+1}}(x) - f(x) + \sum_{j=1}^{\nu} [Q_{n_j}(x) + a_{q_j} W_{q_j}(x)] \right\|_{L^p(E)} < 2^{-4(\nu+1)} + 2^{-4n_{\nu+1}}. \tag{27}
\]

After the selection of functions (see (18) and (23)) we choose
\[
q_{\nu+1} = \min \left\{ \bigcup_{s=1}^{\nu+1} \left\{ k \right\} k=m_{n_{s-1}} \right\} \cup \{q_1, \ldots, q_\nu\}.
\]

By (27) we obtain
\[
\left\| f(x) - \sum_{j=1}^{\nu+1} [Q_{n_j}(x) + a_{q_{\nu+1}} W_{q_{\nu+1}}(x)] \right\|_{L^{p_{\nu+1}}(E)} \leq 2^{-2(\nu+1)} + a_{q_{\nu+1}}. \tag{28}
\]

Thus, by induction from (22) we get some rearranged series
\[
\sum_{k=1}^{\infty} \delta_{\sigma(k)} a_{\sigma(k)} W_{\sigma(k)}(x) = \sum_{s=1}^{\nu} \sum_{k=m_{n_{s-1}}}^{m_{n_s}-1} \delta_k a_k W_k(x) + a_{q_j} W_{q_j}(x), \tag{29}
\]

whose terms satisfy (25) and (26) for all \(\nu > 1\). This together with (26)–(28) implies, that the series (29) converges to \(f(x)\) in any norm \(L^p(E)\), \(p \geq 1\), i.e. the series (22) is quasi universal in \(L^p_{[0,1]}\) with respect to the rearrangements. \(\square\)

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