PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2016, № 1, p. 22–29

Mathematics

ON QUASI-UNIVERSAL WALSH SERIES IN $L^p_{[0,1]}$, $p \in [1,2]$

R. G. MELIKBEKYAN *

Chair of Higher Mathematics, Faculty of Physics YSU, Armenia

Let the sequence $\{a_k\}_{k=1}^{\infty}$, $a_k \searrow 0$ with $\{a_k\}_{k=1}^{\infty} \notin l_2$, and Walsh system $\{W_k(x)\}_{k=0}^{\infty}$ be given. Then for any $\varepsilon > 0$ there exists a measurable set $E \subset [0,1]$ with measure $|E| > 1-\varepsilon$ and numbers $\delta_k = \pm 1, 0$ such that for any $p \in [1,2]$ and each function $f(x) \in L^p(E)$ there exists a rearrangement $k \to \sigma(k)$ such that the series $\sum\limits_{k=1}^{\infty} \delta_{\sigma(k)} a_{\sigma(k)} W_{\sigma(k)}(x)$ converges to $f(x)$ in the norm of $L^p(E)$.

MSC2010: 42C20.

***Keywords*:** Walsh system, quasi universal series.

Introduction. A series

$$\sum_{k=1}^{\infty} f_k(x), \{f_n(x)\}_{n=1}^{\infty} \in L^p[0,1), p \in [1,\infty), \tag{A}$$

is said to be quasi universal in $L^p[0,1)$ with respect to rearrangements, if for any $\varepsilon > 0$ there exists a measurable set $E \subset [0,1]$ with measure $|E| > 1-\varepsilon$ such that for any $p \geq 1$ and each function $f(x) \in L^p(E)$ there exists a rearrangement $k \to \sigma(k)$ such that the series $\sum\limits_{k=1}^{\infty} f_{\sigma}(k)(x)$ converges to $f(x)$ in the norm of $L^p(E)$.

The question of existence of various types of universal series in the sense of almost everywhere or in measure convergence have been considered in [1–8]. The first trigonometric series universal in the usual sense in the class of all measurable functions for convergence almost everywhere has been constructed by Men'shov [1]. This result was extended by Talalyan [2] to arbitrary orthonormal complete systems. Also was established, that if $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in [0,1]$, is an arbitrary complete orthonormal system, then there exists a series $\sum\limits_{k=1}^{\infty} a_k \varphi_k(x)$, which is universal with respect

* E-mail: melikbekyan@yahoo.com

to the convergence in measure of the partial series in the class of all measurable functions on $[0,1]$ [3].

In [4, 5] the following theorems were proved:

Theorem 1. For any complete orthonormal system $\{\varphi_n(x)\}_{n=1}^{\infty}$ there exists a series $\sum_{k=1}^{\infty} b_k\varphi_k(x)$ with $\sum_{k=1}^{\infty} |b_k|^r < \infty$ for any $r > 2$, which is quasi universal in all spaces $L^p[0,1]$ for $p \in [1,2)$ simultaneously with respect to both rearrangements and partial series.

Theorem 2. Let the sequence $\{a_k\}_{k=1}^{\infty}$, $a_k \searrow 0$ with $\{a_k\}_{k=1}^{\infty} \notin l_2$, and the Walsh system $\{W_k(x)\}_{k=0}^{\infty}$ be given. Then there exist numbers $\delta_k = \pm 1$ such that the series $\sum_{k=1}^{\infty} \delta_k a_k W_k(x)$ is universal with respect to a.e. convergence of the partial series in the class of all measurable a.e. finite functions on $[0,1]$.

In this paper we prove the following theorem.

Theorem 3. Let the sequence $\{a_k\}_{k=1}^{\infty}$, $a_k \searrow 0$ with $\{a_k\}_{k=1}^{\infty} \notin l_2$, and the Walsh system $\{W_k(x)\}_{k=0}^{\infty}$ be given. Then for any $\varepsilon > 0$ there exist a measurable set $E \subset [0,1]$ with measure $|E| > 1-\varepsilon$ and numbers $\delta_k = \pm 1$ such that for any $p \in [1,2]$ and each function $f(x) \in L^p(E)$ there exists a rearrangement $k \to \sigma(k)$ such that the series $\sum_{k=1}^{\infty} \delta_{\sigma(k)} a_{\sigma(k)} W_{\sigma(k)}(x)$ converges to $f(x)$ in the norm of $L^p(E)$.

First we show that for any $p \geq 2$ ($p \geq 1$) and for any orthonormal (bounded orthonormal) system $\{\varphi_n(x)\}$ there is no series, that is universal either with respect to rearrangements or signs in $L^p[0,1]$ (for $p > 2$ we assume that $\varphi_n(x) \in L^p[0,1]$, $n = 1,2,...$).

Indeed, if for some $p_0 \geq 2$ ($p_0 \geq 1$) and for some orthonormal (bounded orthonormal) system $\{\varphi_n(x)\}$ there exists a series (1^0), which is universal in $L^{p_0}_{[0,1]}$ with respect to rearrangements, then according to Eq. (A) for $(|a_1|+1)\varphi_1(x)$ there exists a rearrangement $k \to \sigma(k)$ such that

$$\lim_{m\to\infty} \left\| \sum_{k=1}^{m} a_{\sigma(k)}\varphi_{\sigma(k)}(x) - (|a_1|+1)\varphi_1(x) \right\|_{p_0} = 0,$$

or since $\varphi_1(x) \in L^{q_0}[0,1]$ $\left(\frac{1}{q_0} + \frac{1}{p_0} = 1 \text{ for } p_0 > 1 \text{ and } q_0 = \infty \text{ for } p_0 = 1 \right)$, for the first case we have $a_1 = 1 + |a_1|$, while for the second case

$$1 + |a_1| = \begin{cases} a_1 & \text{for} \quad n_1 = 1, \\ 0 & \text{for} \quad n_1 > 1. \end{cases}$$

This contradiction completes the Proof.

Proof of Basic Lemmas. We shall use the following lemma of the paper [5].

Lemma 1. Let the sequence $\{a_k\}_{k=1}^{\infty}$, $a_k \searrow 0$ with $\{a_k\}_{k=1}^{\infty} \notin l_2$, the numbers $n_0 > 1$ ($n_0 \in \mathbb{N}$), $\gamma \neq 0$, $\varepsilon > 0$, $\delta > 0$ and the interval of the form $\Delta = \Delta_m^{(k)} = \left[\frac{k-1}{2^m}, \frac{k}{2^m} \right)$, $k \in [1, 2^m]$, be given. Then, there exist a measurable set

$E \subset \Delta$ and polynomials $H(x)$, $Q(x)$ in Walsh system of the form

$$H(x) = \sum_{k=N_0}^{N} b_k W_k(x), \quad Q(x) = \sum_{k=N_0}^{N} \delta_k a_k W_k(x),$$

satisfying the conditions:

a) $\delta_k = \pm 1, 0$;

b) $|E| = |\Delta|(1-\varepsilon)$;

c) $H(x) = 0,\ x \in [0,1] \setminus \Delta$;

d) $|Q(x) - \gamma| < \delta,\ x \in E$;

e) $\max\limits_{N_0 \le n < N} \left| \sum\limits_{k=N_0}^{n} b_k W_k(x) \right| < C \dfrac{|\gamma|}{\varepsilon} + \delta,\ x \in \Delta$ (C is an absolute constant);

f) $\int_0^1 |Q(x) - H(x)|^2 dx < \varepsilon \delta^2 |\Delta|$;

g) $\max\limits_{N_0 \le n < N_\nu} \left| \sum\limits_{k=N_0}^{n} b_k W_k(x) \right| < \delta, x \in [0,1] \setminus \Delta$.

Using Lemma 1, we prove the following lemma, which is the basic tool in the Proof of Theorem 3.

Lemma 2. Let the sequence $\{a_k\}_{k=1}^{\infty}$, $a_k \searrow 0$ with $\{a_k\}_{k=1}^{\infty} \notin l_2$, be given. Then, for any numbers $N_0 \in \mathbb{N}$, $0 < \delta$, $\varepsilon < 1, p \in [1,2]$ and for each polynomial $f(x)$ in Walsh system with $|f| > 0$ one can find a measurable set $E \subset [0,1]$ and a polynomial in Walsh system of the form

$$Q(x) = \sum_{k=N_0}^{N} \delta_k a_k W_k(x) \ \text{ when } \ \delta_k = \pm 1$$

satisfying the conditions:

1) $|E| = 1 - \varepsilon$;

2) $\left(\int_E |Q(x) - f(x)|^2 dx \right)^{1/2} < \varepsilon$;

3) $\max\limits_{N_0 \le n < N} \left(\int_E \left| \sum\limits_{k=N_0}^{n} \delta_k a_k W_k(x) \right|^p dx \right)^{1/p} \le \left(\int_E |f(x)|^p dx \right)^{1/p} + \varepsilon, \forall p \in [1,2]$.

Proof of Lemma 2. Let

$$f(x) = \sum_{\nu=1}^{\mu_0} \gamma_\nu \chi_{\Delta_\nu}(x), \ \ [0,1) = \bigcup_{\nu=1}^{\mu_0} \Delta_\nu, \tag{1}$$

where Δ_ν, $\nu = 1,2,...,\mu_0$, are disjoint diadic intervals (χ_Δ is characteristic function of Δ). We choose a number $\beta > 0$ such that

$$\max\left\{ \beta, \beta \sum_{\nu=1}^{\mu_0} |\Delta_\nu|^{1/2} \right\} < \min\left\{ \frac{\delta}{4}; C\frac{\min|f|}{\varepsilon}; \left(\int_0^1 |f|^p dx \right)^{1/p} \right\}. \tag{2}$$

Consecutively applying Lemma 1, one can define sets $E_\nu \subset \Delta_\nu$ for $1 \le \nu \le \mu_0$ and polynomials

$$H_\nu(x) = \sum_{k=N_{\nu-1}}^{N_\nu - 1} b_k W_k(x), \tag{3}$$

$$Q_\nu(x) = \sum_{k=N_{\nu-1}}^{N_\nu - 1} \delta_k a_k W_k(x),\ \delta_k = \pm 1, \tag{4}$$

where $N_0 = n_0$, satisfying the conditions:

$$|E_\nu| = |\Delta_\nu|(1-\varepsilon), \tag{5}$$

$$H_\nu(x) = 0,\ x \in [0,1] \setminus \Delta_\nu, \tag{6}$$

$$|H_\nu(x) - \gamma_\nu| < \beta,\ x \in E_\nu, \tag{7}$$

$$\max_{N_{\nu-1} \le n < N_\nu} \left| \sum_{k=N_{\nu-1}}^{n} b_k W_k(x) \right| < C \frac{|\gamma_\nu|}{\varepsilon} + \beta,\ x \in \Delta_\nu\ (C > 1), \tag{8}$$

C is an absolute constant,

$$\int_0^1 |Q_\nu(x) - H_\nu(x)|^2 dx < \varepsilon \beta^2 |\Delta_\nu|, \tag{9}$$

$$\max_{N_{\nu-1} \le n < N_\nu} \left| \sum_{k=N_{\nu-1}}^{n} b_k W_k(x) \right| < \beta,\ x \in [0,1] \setminus \Delta_\nu. \tag{10}$$

We put

$$E = \bigcup_{\nu=1}^{\mu_0} E_\nu, \tag{11}$$

$$Q(x) = \sum_{\nu=1}^{\mu_0} Q_\nu(x) = \sum_{\nu=1}^{\mu_0} \sum_{k=N_{\nu-1}}^{N_\nu - 1} \delta_k a_k W_k(x) = \sum_{k=N_0}^{N} \delta_k a_k W_k(x), \tag{12}$$

$$H(x) = \sum_{\nu=1}^{\mu_0} H_\nu\ (x) = \sum_{\nu=1}^{\mu_0} \sum_{k=N_{\nu-1}}^{N_\nu - 1} a_k W_k(x) = \sum_{k=N_0}^{N} a_k W_k(x). \tag{13}$$

From (5) and (11) it follows that $|E| = 1 - \varepsilon$. Using equations (3), (4), (12), (13) and (9), we have

$$\left(\int_0^1 |Q(x) - H(x)|^2 dx \right)^{1/2} \le \sum_{\nu=1}^{\mu_0} \left(\int_0^1 |Q_\nu(x) - H_\nu(x)|^2 dx \right)^{1/2} \le$$

$$\le \beta \sum_{\nu=1}^{\mu_0} (\varepsilon |\Delta_\nu|)^{1/2} < \min \left\{ \frac{\delta}{4}; \left(\int_0^1 |f|^p dx \right)^{1/p} \right\}. \tag{14}$$

Then, according to (1), (2) and (6), (7), (9), we get

$$\left(\int_E |H(x)-f(x)|^2dx\right)^{1/2} \le \sum_{\nu=1}^{\mu_0}\left(\int_{E_\nu}|H_\nu(x)-\gamma_\nu|^2dx\right)^{1/2} \le$$

$$\le \sum_{\nu=1}^{\mu_0}\left(\int_{E_\nu}|H_\nu(x)-Q_\nu(x)|^2dx\right)^{1/2} + \sum_{\nu=1}^{\mu_0}\left(\int_{E_\nu}|Q_\nu(x)-\gamma_\nu|^2dx\right)^{1/2} < \quad (15)$$

$$< \beta\sum_{\nu=1}^{\mu_0}(\varepsilon|\Delta_\nu|)^{1/2} + \beta\sum_{\nu=1}^{\mu_0}|\Delta_\nu|^{1/2} < \frac{\delta}{2}.$$

From this and from relations (12)–(14) we conclude that

$$\left(\int_E |Q(x)-f(x)|^2dx\right)^{1/2} \le \left(\int_0^1 |Q(x)-H(x)|^2dx\right)^{1/2} +$$

$$+\left(\int_E |H(x)-f(x)|^2dx\right)^{1/2} < \delta,$$

which proves the condition 3) of the Lemma.

If $n\in[N_0,N]$, then for some $1\le\nu\le\mu_0$ and $n\in[N_{\nu-1},N_\nu)$ from (14) we get

$$\sum_{k=N_0}^{n}\delta_k a_k W_k(x) = \sum_{j=1}^{\nu-1}\sum_{k=N_{j-1}}^{N_j-1}\delta_k a_k W_k(x) + \sum_{k=N_{\nu-1}}^{n}\delta_k a_k W_k(x), \quad (16)$$

From this and (4), (6), (8), (13), (14), for any $p\in[1,2]$ we obtain

$$\left(\int_E\left|\sum_{k=N_0}^{n}\delta_k a_k W_k(x)\right|^p dx\right)^{1/p} \le$$

$$\le \sum_{J=1}^{\nu-1}\left(\int_E\left|Q_j(x)-H_j(x)\right|^p dx\right)^{1/p} + \sum_{j=1}^{\nu-1}\left(\int_E\left|H_j(x)\right|^p dx\right)^{1/p} +$$

$$+\left(\int_E|\sum_{k=N_{\nu-1}}^{n}(\delta_k a_k - a_k)W_k(x)|^p dx\right)^{1/p} + \left(\int_E|\sum_{k=N_{\nu-1}}^{n}a_k W_k(x)|^p dx\right)^{1/p} \le$$

$$\le \beta\sum_{J=1}^{\nu-1}(\varepsilon|\Delta_\nu|)^{1/2} + 2\left(\sum_{\nu=1}^{\mu_0}(|\gamma_\nu+\beta|)^p|\Delta_\nu|\right)^{1/p} + \left(C\frac{|\gamma_\nu|}{\varepsilon}+\beta\right)(|\Delta_\nu|)^{1/p} \le$$

$$\le 2\left(\int_E |f(x)|^p dx\right)^{1/p} + \varepsilon. \quad \square$$

Proof of Theorem 3. Let

$$\left\{f_k(x)\right\}_{k=1}^{\infty} \quad (17)$$

be the system of all algebraic polynomials with rational coefficients and the sequence $\{a_k\}_{k=1}^{\infty}$, $a_k \searrow 0$ with $\{a_k\}_{k=1}^{\infty}\notin l_2$, and $\varepsilon\in(0,1)$ be given.

Applying Lemma 2 when $f = f_n$ from (17), we determine a sequence of measurable sets $\{E_n\}$ and a sequence of polynomials

$$Q_n(x) = \sum_{k=m_{n-1}}^{m_n-1} \delta_k a_k W_k(x), \ \ \delta_k = \pm 1, 0, \ m_0 = 2\ , \ m_n > m_{n-1}, \qquad (18)$$

$$|E_n| > 1 - 2^{-2n}, \qquad n \geq 1, \qquad (19)$$

$$\left(\int_{E_n} |Q_n(x) - f_n(x)|^2 dx \right)^{1/2} < 2^{-4n}, \qquad (20)$$

$$\max_{m_{n-1} \leq s \leq m_n} \left[\int_{E_n} \left| \sum_{k=m_{n-1}}^{s} \delta_k a_k W_k(x) \right|^p dx \right]^{1/p} \leq$$

$$\leq \left(\int_{E_n} |f_n(x)|^p\, dx \right)^{1/p} + 2^{-2n}, \ \ p \in [1,2]. \qquad (21)$$

We put

$$\sum_{k=1}^{\infty} \delta_k a_k W_k(x) = \sum_{n=1}^{\infty} \left(\sum_{k=m_{n-1}}^{m_n-1} \delta_k a_k W_k(x) \right), \qquad (22)$$

$$E = \bigcap_{\nu=n_0}^{\infty} E_\nu, \qquad (23)$$

where n_0 is the integer part of $\log_{1/2} \varepsilon$. From (19) and (23) we get $|E| > 1 - \varepsilon$.

Let $f(x) \in L^p(E),\ p \in [1,2]$. Consider a function $f_{n_1}(x)\ (n_1 > n_0)$ from (2) satisfying

$$||f(x) - f_{n_1}(x)||_{L^p(E)} = \left(\int_E |f(x) - f_{n_1}(x)|^p\, dx \right)^{1/p} < 2^{-4(1+1)}.$$

From this and (20)

$$\|f(x) - [Q_{n_1}(x) + a_1 W_1(x)]\|_{L^{p_1}(E)} < 2^{-2n_1} + 2^{-8} + a_1.$$

Assume that the members

$$\left\{ \left\{ \delta_j = \pm 1, k_j \right\}_{j=m_{n_s-1}}^{m_{n_s}-1}, q_s \right\}_{s=1}^{\nu}$$

and functions

$$\left\{ \delta_k a_k W_k(x) \right\}_{k=m_{n_s-1}}^{m_{n_s}-1},\ a_{q_s} W_{q_s}(x),\ \ s = 1, 2, \ldots, \nu,$$

of the series (22) or (23) are chosen to satisfy

$$q_\nu = \min \left\{ \bigcup_{s=1}^{\nu} \left\{ k \right\}_{k=m_{n_s-1}}^{m_{n_s}-1} \right\} \cup \{q_1, \ldots, q_{\nu-1}\}.$$

$$\left\| f(x) - \sum_{j=1}^{\nu} \left[Q_{n_j}(x) + a_{q_j} W_{q_j}(x) \right] \right\|_{L^p(E)} \leq 2^{-2n_\nu} + 2^{-4(\nu+1)} + a_{s_\nu}\ , \qquad (24)$$

$$\max_{m_{n_\nu-1}\le m\le m_{n_\nu}}\left\|\sum_{k=m_{n_\nu-1}}^{m}\delta_j a_k W_k(x)\right\|_{L^p(E)} < 2^{-\nu}+a_{q_\nu}\ ,\ \forall p\in[1,2]\ . \tag{25}$$

We choose a natural number $n_{\nu+1}>n_\nu$ and a function $f_{n_{\nu+1}}(x)$ from (2) satisfying

$$\left\|f_{n_{\nu+1}}(x)-f(x)+\sum_{j=1}^{\nu}\left[Q_{n_j}(x)+a_{q_j}W_{q_j}(x)\right]\right\|_{L^p(E)} < 2^{-4(\nu+1)}.$$

From this and from (25) we obtain

$$\left\|f_{n_{\nu+1}}(x)\right\|_{L^p(E)} \le 2^{-2(\nu+1)}+a_{q_\nu}\ .$$

Therefore, by (18), (21) and (23) we have

$$\max_{m_{n_{\nu+1}-1}\le N\le m_{n_{\nu+1}}}\left\|\sum_{k=m_{n_{\nu+1}-1}}^{N}\delta_k a_k W_k(x)\right\|_{L^p(E)} < 2^{-(\nu+1)}+a_{q_\nu}\ , \tag{26}$$

$$\left\|Q_{n_{\nu+1}}(x)-f(x)+\sum_{j=1}^{\nu}\left[Q_{n_j}(x)+a_{q_j}W_{q_j}(x)\right]\right\|_{L^p(E)} < 2^{-4(\nu+1)}+2^{-4n_{\nu+1}}. \tag{27}$$

After the selection of functions (see (18) and (23)) we choose

$$q_{\nu+1}=\min\left\{\bigcup_{s=1}^{\nu+1}\left\{k\right\}_{k=m_{n_s-1}}^{m_{n_s}-1}\right\}\cup\{q_1,\dots,q_\nu\}.$$

By (27) we obtain

$$\left\|f(x)-\sum_{j=1}^{\nu+1}\left[Q_{n_j}(x)+a_{q_{\nu+1}}W_{q_{\nu+1}}(x)\right]\right\|_{L^{p_{\nu+1}}(E)} \le 2^{-2(\nu+1)}+a_{q_{\nu+1}}\ . \tag{28}$$

Thus, by induction from (22) we get some rearranged series

$$\sum_{k=1}^{\infty}\delta_{\sigma(k)}a_{\sigma(k)}W_{\sigma(k)}(x)=\sum_{s=1}^{\infty}\left(\sum_{k=m_{n_s-1}}^{m_{n_s}-1}\delta_k a_k W_k(x)+a_{q_s}W_{q_s}(x)\right)\ , \tag{29}$$

whose terms satisfy (25) and (26) for all $\nu>1$. This together with (26)–(28) implies, that the series (29) converges to $f(x)$ in any norm $L^p(E),\ p\ge 1$, i.e. the series (22) is quasi universal in $L^p_{[0,1]}$ with respect to the rearrangements. □

Received 11.12.2015

REFERENCES

1. **Men'shov D.E.** On the Partial Sums of Trigonometric Series. // Mat. Sbornik, 1947, v. 20, № 2, p. 197–238 (in Russian).
2. **Talalyan A.A.** On the Convergence Almost Everywhere of the Subsequences of Partial Sums of General Orthogonal Series. // Izv. AN Arm. SSR. Matematika, 1975, v. 10, № 3, p. 17–34 (in Russian).
3. **Talalyan A.A.** Representation of Measurable Functions by Series. // UMN, 1960, v. 15, № 5, p. 567–604 (in Russian).
4. **Grigorian M.G.** On the Representation of Functions by Orthogonal Series in Weighted L^p Spaces. // Studia Math., 1999, v. 134, № 3, p. 207–216.
5. **Gevorkyan G.G., Navasardyan K.A.** On the Walsh Series with Monotone Coeffisients. // Izv. RAN, Math., 1999, v. 63.2.1, p. 41–60.
6. **Grigorian M.G.** On the Systems Universal in $L^p[0,1]$, $1 \leq p < 2$. // Izv. Vuzov. Matematika, 2000, v. 456, № 5, p. 19–22.
7. **Grigorian M.G.** Universal or Quasi Universal in $L^p_{[0,1]}$ Orthogonal Series. // Journal of Contemporary Mathematical Analysis, 2000, v. 35, № 4, p. 44–64.
8. **Navasardyan K.A.** On the Walsh Series with Monotone Coeffisients. // Izv. NAN Armenia, 2007, v. 42, № 5, p. 51–64.