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Mathematics

ON A CONJECTURE IN BIVARIATE INTERPOLATION

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Denote the space of all bivariate polynomials of total degree $\leq n$ by Π_n. We are interested in n-poised sets of nodes with the property that the fundamental polynomial of each node is a product of linear factors. In 1981 M. Gasca and J.I.Maeztu conjectured that every such set contains necessarily $n+1$ collinear nodes. Up to now this had been confirmed for degrees $n \leq 5$. Here we bring a simple and short proof of the conjecture for $n = 4$.

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1. Introduction. Denote by Π_n the space of bivariate polynomials of total degree not greater than n. We have

$$N := \dim \Pi_n = \binom{n+2}{2}.$$

We call a set $\mathcal{X}_s = \{(x_1,y_1),(x_2,y_2),\ldots,(x_s,y_s)\}$ of distinct nodes n-poised, if for any data $\{c_1,\ldots,c_s\}$ there exists a unique polynomial $p \in \Pi_n$ satisfying the conditions

$$p(x_i,y_i) = c_i, \qquad i = 1,2,\ldots s. \tag{1.1}$$

A necessary condition of n-poisedness is $s = N$.

A polynomial $p \in \Pi_n$ is called an n-fundamental polynomial for a node $A = (x_k,y_k) \in \mathcal{X}_s$, if

$$p(x_k,y_k) = 1 \quad \text{and} \quad p(x_i,y_i) = 0 \quad \text{for all} \quad 1 \leq i \leq s,\ i \neq k.$$

We denote the n-fundamental polynomial of $A \in \mathcal{X}_s$ by $p_A^\star$.

We shall use the same letter, say ℓ, to denote the polynomial $\ell \in \Pi_1$ and the line with an equation $\ell(x,y) = 0$.

Now consider an n-poised set $\mathcal{X} = \mathcal{X}_N$. We say, that a node $A \in \mathcal{X}$ uses a line ℓ, if ℓ is a factor of the fundamental polynomial $p_A^\star$. We are going to use the following well known

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P r o p o s i t i o n . Suppose that ℓ is a line and $\mathcal{X}$ is a set of $n+1$ nodes lying in ℓ. Then for any polynomial $p \in \Pi_n$ vanishing on $\mathcal{X}$ we have

$$p = \ell r, \quad \text{where} \quad r \in \Pi_{n-1}.$$

It follows from the Proposition that at most $n+1$ n-independent nodes can lie in a line.

We will make use of a special case of Cayley–Bacharach theorem (see, eg., [1], Th. CB4 [2], Prop. 4.1):

T h e o r e m 1 . 1 . Assume that the three lines ℓ_1, ℓ_2, ℓ_3 intersect another three lines $\ell_1^{'}, \ell_2^{'}, \ell_3^{'}$ at nine different points. If a polynomial $p \in \Pi_3$ vanishes at any eight intersection points, then it vanishes at all nine points.

2. The Gasca–Maeztu Conjecture. Here we consider a special type of n-poised sets.

D e f i n i t i o n . We call an n-poised set $\mathcal{X}$ GC_n-set, if each node $A \in \mathcal{X}$ has an n-fundamental polynomial which is a product of n linear factors.

Since the fundamental polynomial of an n-poised set is unique each of these lines passes through at least two nodes from $\mathcal{X}$ not belonging to the other lines.

Next we bring the Gasca–Maeztu conjecture:

C o n j e c t u r e [3]. Any GC_n-set contains $n+1$ collinear nodes.

So far the conjecture was proved for the values $n \leq 5$ (see [4]). In the case $n = 4$ this reduces to the following:

T h e o r e m 2 . 1 . Any GC_4-set $\mathcal{X}$ of 15 nodes contains five collinear nodes.

To prove this, we shall assume from now that:

$$\textit{The set } \mathcal{X} \textit{ is a } GC_4\textit{-set, which does not contain five collinear nodes,} \tag{2.1}$$

in order to derive a contradiction.

For each node $A \in \mathcal{X}$ the 4-fundamental polynomial is a product of four linear factors. In view of assumption (2.1) the 14 nodes of $\mathcal{X} \setminus \{A\}$ are distributed in the four lines used by A in two possible ways: $4+4+4+2$ or $4+4+3+3$. Accordingly, we can represent $p_A^\star$ in two forms:

$$p_A^\star = \ell_{=4}\ell_{=4}^{'}\ell_{=4}^{''}\ell_{\geq 2}, \qquad p_A^\star = \ell_{=4}\ell_{=4}^{'}\ell_{\geq 3}\ell_{\geq 3}^{'}. \tag{2.2}$$

The lines with $= k$ in the subscript are called k-node lines and pass through exactly k nodes. The lines with $\geq k$ in the subscript pass through k nodes and possibly also through some other, already counted nodes, which are the intersection points with the other lines.

2.1. Lines Used by Several Points. We start with two lemmas from [5] (see Lemmas 2.5 and 2.6). For the sake of completeness we state their proofs.

L e m m a 2 . 1 . Any 2 or 3-node line can be used by at most one node of $\mathcal{X}$.

P r o o f . Assume to the contrary that $\tilde{\ell}$ is a 2 or 3-node line used by two points $A, B \in \mathcal{X}$. Consider the fundamental polynomial $p_A^\star$ of A. It consists of the line $\tilde{\ell}$ and three more lines, which contain the remaining ≥ 11 nodes of $\mathcal{X} \setminus (\tilde{\ell} \cup \{A\})$ including B. Since there is no 5-node line, we get

$$p_A^\star = \tilde{\ell}\ell_{=4}\ell_{=4}^{'}\ell_{\geq 3}.$$

First suppose that B belongs to one of the 4-node lines, say to $\ell_{=4}^{'}$. We have that

$$p_B^\star = \tilde{\ell} q, \text{ where } q \in \Pi_3.$$

Notice that q vanishes at 4 nodes of $\ell_{=4}$, therefore, according to Proposition

$$p_B^\star = \tilde{\ell}\ell_{=4}r, \text{ where } r \in \Pi_2.$$

Now r vanishes at 3 nodes of $\ell'_{=4}$ (i.e. except B). Therefore, again from Proposition we get that r vanishes at all points of $\ell'_{=4}$ including B. Hence $p_B^\star$ vanishes at B, which is a contradiction.

Assume that B belongs to the line $\ell_{\geq 3}$. Then q vanishes at 4 nodes of $\ell_{=4}$, 4 (≥ 3) nodes of $\ell'_{=4}$ and at least 2 nodes of $\ell_{\geq 3}$. Therefore, we get from Proposition as above

$$p_B^\star = \ell_{=4}\ell'_{=4}\ell_{\geq 3}\ell, \quad \text{where} \quad \ell \in \Pi_1.$$

Hence again $p_B^\star$ vanishes at B, which is a contradiction. □

Lemma 2.2. Any 4-node line $\tilde{\ell}$ can be used by at most three nodes of $\mathcal{X}$. If three nodes use $\tilde{\ell}$, then they share two more 4-node lines.

Proof. Assume that $\tilde{\ell} = \tilde{\ell}_{=4}$ is a 4-node line and is used by two nodes $A, B \in \mathcal{X}$. Consider the fundamental polynomial $p_A^\star$ of $A \in \mathcal{X}$. It consists of the line $\tilde{\ell}$ and three more used lines containing the remaining 10 nodes of $\mathcal{X} \setminus (\ell \cup \{A\})$ including B. Now there are two possibilities:

$$p_A^\star = \tilde{\ell}\ell_{=4}\ell'_{=4}\ell_{\geq 2}, \tag{2.3}$$

$$p_A^\star = \tilde{\ell}\ell_{=4}\ell_{\geq 3}\ell'_{\geq 3}. \tag{2.4}$$

Consider first the case (2.3). We readily get, as in the proof of Lemma 2.1, that B does not belong to any of the 4-node lines. Hence B must belong to the line $\ell_{\geq 2}$. The same statement is true also for the third point C that may use $\tilde{\ell}$. Clearly, there is no room for the fourth node in $\ell_{\geq 2}$. Thus, Lemma is proved in this case. Notice that in this case $p_B^\star$ ($p_A^\star$) uses the lines $\tilde{\ell}, \ell_{=4}, \ell'_{=4}$ and the line ℓ_{AC} (ℓ_{AB}), where ℓ_{AC} is the line passing through A and C. Therefore, the nodes A, B, C share three 4-node lines.

Now consider the case when $p_A^\star$ is given by (2.4). As above B does not belong to any of the 3-node lines. Hence, B must belong to the line $\ell_{=4}$. We have also

$$p_B^\star = \tilde{\ell}\alpha_{=4}\alpha_{\geq 3}\alpha'_{\geq 3}. \tag{2.5}$$

Moreover we have that the node A in its turn belongs to the line $\alpha_{=4}$.

Now let us verify that the nodes A and B do not share any line except $\tilde{\ell}$. Indeed, $\ell_{=4}$ and $\alpha_{=4}$ are the only lines containing B and A respectively. Thus, they are not among the possibly coinciding lines. Without loss of generality assume to the contrary that $\ell'_{\geq 3} \equiv \alpha'_{\geq 3}$.

Thus, we have

$$p_B^\star = \tilde{\ell}\ell'_{\geq 3}q, \text{ where } q \in \Pi_2.$$

Then, in view of Eq. (2.4), q vanishes at 3 nodes of $\ell_{\geq 3}$ and at least at 2 nodes of $\ell_{=4}$ (i.e. except B and a possible secondary node of $\ell'_{\geq 3}$). Therefore, from Proposition we get that q vanishes at all points of the lines $\ell_{=4}$ and $\ell_{\geq 3}$ including B. Hence $p_B^\star$ vanishes at B, which is a contradiction.

Thus the triples of the used lines $\ell_{=4}$, $\ell'_{\geq 3}$, $\ell_{\geq 3}$ and $\alpha_{=4}$, $\alpha'_{\geq 3}$, $\alpha_{\geq 3}$ intersect at exactly 9 nodes of $\mathcal{I} := \mathcal{X} \setminus [\tilde{\ell} \cup \{A, B\}]$.

If a third node C uses $\tilde{\ell}$, then we have that $C \in \mathcal{I}$ and p_C^* must vanish at eight nodes of $\mathcal{I}$, but, by Theorem 1.1, p_C^* vanishes at C too, which is a contradiction. Therefore, the 4-node line in this case can be used at most twice. □

Corollary. Suppose ℓ is a 4-node line and three nodes $A,B,C \in \mathcal{X} \setminus \ell$ use a line $\tilde{\ell}$, Then l is among the three lines used by A,B and C.

Indeed, the 12 nodes of $\mathcal{X} \setminus \{A,B,C\}$ lie in the three used lines.

With the following Lemma we strengthen the Lemma 2 in [6].

Lemma 2.3. If a node $A \in \mathcal{X}$ uses a 4-node line ℓ, then there are two more nodes in $\mathcal{X}$ using it.

Proof. Denote the four lines joining the node A with the 4 nodes on ℓ by $\ell_1, \ell_2, \ell_3, \ell_4$. Consider any node from $\mathcal{X} \setminus (\ell \cup \{A\})$. The 4 lines used by it pass through five nodes in $\ell \cup \{A\}$. Therefore, one of lines passes through 2 nodes of the five and, therefore, coincides with one of the lines $\ell_1, \ell_2, \ell_3, \ell_4$ or ℓ. Thus, each of 11 nodes of $\mathcal{X} \setminus \ell$ uses one of the lines $\ell, \ell_1, \ell_2, \ell_3, \ell_4$. Hence, there are at least three nodes in $\mathcal{X} \setminus \ell$ using the same line $\tilde{\ell}$ among the mentioned 5. In view of Corollary $\tilde{\ell}$ is used by the same triple of nodes. Node A is among the three nodes from Lemma 2.2. □

Remark. Notice that $\tilde{\ell} \equiv \ell$. Indeed, $\tilde{\ell}$ is used by A and, therefore, it cannot coincide with any $\ell_i, i = 1,2,3,4$, Also, each of the lines $\ell, \ell_1, \ell_2, \ell_3, \ell_4$ is used by exactly 2 nodes from 10 of $\mathcal{X} \setminus (\ell \cup \{A\})$. Therefore, in view of Lemma 2.2, each of the lines $\ell_1, \ell_2, \ell_3, \ell_4$ is a 4-node line.

2.2. Proof of Theorem 2.1. It follows from Lemmas 2.2 and 2.3, that all the fundamental polynomials of $\mathcal{X}$ have the form $4+4+4+2$, i.e. we have the first case of Eq. (2.2).

Suppose a node A uses a 4-node line ℓ and the four lines $\ell_1, \ell_2, \ell_3, \ell_4$ pass through A and the four nodes in ℓ, as in the proof of Lemma 2.3. The line ℓ is used by two more nodes $B, C \notin \ell$ (Lemma 2.3). The nodes A,B,C share two more 4-node lines, which we denote by $\ell^{'}$ and $\ell^{''}$. Let us verify that the nodes B and C do not lie in the lines $\ell_1, \ell_2, \ell_3, \ell_4$. Indeed, suppose conversely, say, the node B is in ℓ_1. Then, in view of Remark, C does not use any of the 4-node lines $\ell_1, \ell_2, \ell_3, \ell_4$, while from the Proof of Lemma 2.2 we have that C uses the line passing through A and B and thus coinciding with ℓ_1.

Therefore, 12 nodes of $\mathcal{X} \setminus \{A,B,C\}$ belong to the lines $\ell_1, \ell_2, \ell_3, \ell_4$. On the other side, these 12 nodes belong to the lines $\ell, \ell^{'}$ and $\ell^{''}$.

Now, we may conclude that 12 nodes of $\mathcal{X} \setminus \{A,B,C\}$ are the intersection points of the 4 lines $\ell_1, \ell_2, \ell_3, \ell_4$ with the 3 lines $\ell, \ell^{'}$ and $\ell^{''}$.

Finally consider the polynomial $p = \ell_1 \ell_2 \ell_3 \ell_4$. As the fourth degree polynomial p vanishes at all the nodes, but B and C, it should be a linear combination of the fundamental polynomials of these two nodes. Both these fundamental polynomials vanish on the lines $\ell, \ell^{'}, \ell^{''}$, so this should be true also for p, which is a contradiction.

□

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