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TRANSMISSION OF NARROW SOUND BEAMS IN NONLINEAR ENVIRONMENT

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In the paper we consider the problem of the transmission of limited sound beams. The transmission of such beams is described by a nonlinear partial differential equation. In the paper we solve this equation by the lattice-characteristics method. Some numerical results are obtained for a special case.

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1. Introduction. In the paper we consider the non-linear problem of the limited sound beams transmission. The interest toward this topic is explained by the possibility of its wide practical application. The transmission of such beams is described by a non-linear equation presented in [1,2]. Since of the absence of regular methods of solving of nonlinear partial differential equations it is not possible to obtain the general solution of non-linear acoustics equation. In the paper this equation is solved by the finite difference schemes method [3] in a three dimensional domain.

2. A Numerical Solution of the Limited Narrow Sound Beams Transmission Problem. The equation of non-linear acoustics is convenient to present in non-dimensional variables. In [4] it is brought to the following form:

$$N_0 \frac{\partial^2 \rho^2}{\partial \theta^2} - \frac{\partial^2 \rho}{\partial \theta \partial z} + B \Delta_\perp \rho = 0, \tag{1}$$

where $R = \sqrt{x^2 + y^2}$ and $\Delta_\perp = \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial}{\partial R} \right)$ is the Laplace operator, which affects in the plane orthogonal to the direction of the beam transmission. We will seek for the solution of this equation in the following domain:

$$G_3 \; (0 \leqslant \theta \leqslant 2\pi, R \leqslant R_0, 0 \leqslant z \leqslant Z),$$

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where R_0 and Z are real numbers identifying the boundary of the domain, in which the sound beam spreads.

Let us require that the solution $\rho(\theta, R, z)$ satisfies the following initial and boundary conditions:

$$
\begin{aligned}
&1.\ \rho|_{z=0} = e^{-R^2} \sin\theta; \\
&2.\ \rho|_{R=0} = |\rho|_{R=R_0} = 0; \\
&3.\, \rho(\theta, R, z) \text{ is a } 2\pi\text{-periodic function of } \theta.
\end{aligned}
\tag{2}
$$

Next, let us require that the initial function satisfies the following boundary condition:

$$\int\limits_0^{2\pi} F(\theta, R)\, d\theta = 0, \qquad F(\theta, R) = \exp^{-R^2} \sin\theta.$$

Before solving the problem (1), (2), we first consider the case when $B = 0$. Then we obtain a quasi-linear first degree equation

$$\frac{\partial\rho}{\partial z} - 2N_0\rho\frac{\partial\rho}{\partial\theta} = 0, \tag{3}$$

where ρ is a 2π-periodic function satisfying the initial condition

$$\rho|_{z=0} = F(\theta). \tag{4}$$

To solve this problem, let us make use of the lattice characteristics method [5]. It is known that the Eq. (3) can have non-smooth solutions, which in acoustics and gas dynamics are called shock waves. Concerning the solving Eq. (3) by a numerical method, let us state the following. As far as the solution is a smooth function, to find it one can apply a wide class finite difference schemes methods.

In the case when the solution is not a smooth function, it is necessary to apply some special methods, which take into account the properties of the solution at the point of discontinuity. An example of a such method is one based on the introduction of the concept of artificial viscosity. Being very small, it preserves the characteristic properties of the solution by making it a smooth function. For solving such equation numerically one can apply finite difference scheme of general kind. The method of artificial viscosity has number of drawbacks, one of which is the wave front blurring.

The methods of characteristics and divergent finite difference schemes are solving this type of equations by taking into account the discontinuity. The divergent schemes method is based on the properties of the solution, which follow from the integral conservation law.

Let $F(\theta, z)$ and $\rho(\theta, z)$ be 2π-periodic functions of θ. Let us denote $F(\theta, z)$ simply by $F(\theta)$. Now, by extending periodically the solution and the initial function, consider. The problem in the domain G_2 with $G_2 = (-\infty < \theta < \infty,\ 0 \leqslant z \leqslant Z)$.

Let us introduce the following lines:

$$\frac{\partial\theta}{\partial z} = 2N_0\rho, \tag{5}$$

which we call characteristics for the Eq. (3). Along with the direction of each characteristic $\theta = \theta(z)$ the solution $\rho(\theta, z)$ can be considered as a function $\rho(\theta(z), z)$

of variable z solely. Then the solution is constant along the direction of the characteristic

$$\frac{\partial\rho}{\partial z} = \frac{\partial\rho}{\partial z} + \frac{\partial\rho}{\partial\theta}\frac{\partial\theta}{\partial z} = \frac{\partial\rho}{\partial z} - 2N_0\rho\frac{\partial\rho}{\partial\theta} = 0.$$

By taking into account (5) and the fact that the solution is constant along the direction of the characteristic, we come to the conclusion that the characteristics are the lines $\theta = -2N_0\rho z + \theta_0$, where θ_0 is the x-coordinate of the point $(0, \theta_0)$, from which takes start the characteristic with the slope $-2N_0\rho$ with respect to the z-axis, and ρ is a constant that equals $F(\theta_0) : \rho = F(\theta_0)$.

Thus, the directions of characteristics and the value of the solution, which spreads along the characteristics without a change in the domain G_2, are determined according to the initial function $F(\theta)$.

In the case of a smooth solution the characteristics $\rho(\theta, z)$ do not intersect. In the opposite case each characteristic brings into the point of intersection its value, in which case the solution becomes discontinuous.

If the function $F(\theta)$ is monotonically decreasing along with the increase θ, then the angle between the characteristic and the z-axis is increasing and the characteristics are diverging. But along with the increase of $F(\theta)$ the characteristics are coming closer and intersecting, regardless the fact that the initial function: $F(\theta)$ is smooth. We are arriving in this way to a solution with discontinuities.

Suppose that $\rho(\theta, z)$ is a continuously differentiable solution for the Eq. (3). Let us integrate the Eq. (3) along with some domain $G_2^* \subset G_2$, the boundary of which is Γ^*. By using the Green formula, we obtain

$$\int\limits_{G_2^*} \left(\frac{\partial\rho}{\partial z} - 2N_0\rho\frac{\partial\rho}{\partial\theta}\right) d\theta\, dz = -\int\limits_{\Gamma^*} \rho d\theta + N_0\rho^2 dz,$$

which implies

$$\int\limits_{\Gamma^*} \rho d\theta + N_0\rho^2 dz = 0. \tag{6}$$

Thus the solution of the continuously differentiable Eq. (3) satisfies the integral Eq. (6). Let us extend the set of solutions of the differential Eq. (3) by a piecewise differentiable function and call it a generalized solution. This generalized solution is satisfying the Eq. (6) with arbitrary $\Gamma^* \in G_2$, which is the integral form of the conservation law. Notice that to get a unique solution it is necessary that certain condition, like the entropy increase low, holds on the boundary of the domain.

If on some line $\theta = \theta(z)$ the general solution has a discontinuity by remaining continuous from the right and left of the breaking point then the Eq. (4) implies the following equality

$$\frac{\partial\theta}{\partial z} = -N_0\left(\rho_{\mathrm{left}} + \rho_{\mathrm{right}}\right). \tag{7}$$

In case of other conservation law the slope of the line would be different. To distinguish the generalized solution for the differential equation that prescribes correctly the route of the real process it is necessary to follow physical conservation laws,which yield this equation.

To solve the Eqs. (3),(4) by the difference schemes method, consider the lattice G_2^h, which is a union of two lattices (Fig. 1):

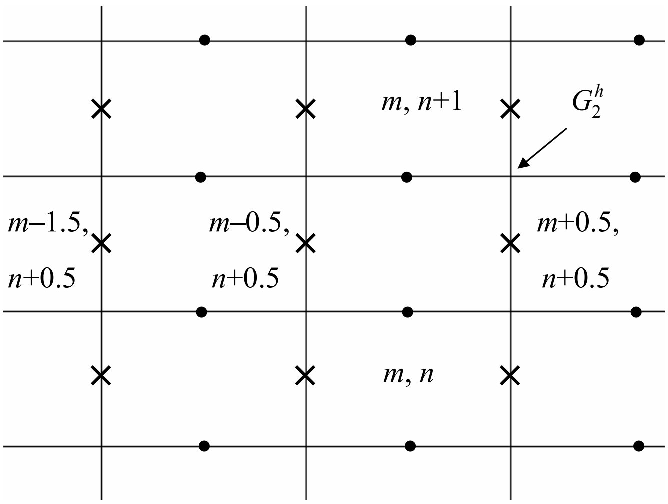

Fig. 1. G_2^h lattice:
$G_2^{h,1}$ $(\theta_m, z_n;\ m = 0, \pm 1, \ldots, N_1;\ n = 0, 1, 2, \ldots, N_2)$,
$G_2^{h,2}$ $(\theta_{m+0.5};\ z_{n+0.5};\ m = 0, \pm 1, \ldots, N_1 - 1;\ n = 0, 1, 2, \ldots, N_2 - 1)$,
$\theta_p = ph_\theta,\ z_q = qh_z,\ h_\theta = 2\pi/N,\ h_z = Z/N_2$.

Let us define lattice functions on $G_2^h : \rho_m^n$ on $G_2^{h,1}$ and $\rho_{m+0.5}^{n+0.5}$ on $G_2^{h,2}$ which are 2π-periodic. The lattice G_2^h is a simplest rectangle with the center $\{\theta_m, z_{m+0.5}\}$ and length of sides equal to h_θ and h_z, which are parallel to the axes θ and z respectively. The boundary of G_2^h is Γ_h.

By replacing the integral preservation law by the mean rectangles quadrature rule, we are arriving to the difference formula

$$\frac{\rho_m^{n+1} - \rho_m^n}{h_z} - N_0 \frac{\left(\rho_{m+0.5}^{n+0.5}\right)^2 - \left(\rho_{m-0.5}^{n+0.5}\right)^2}{h_\theta} = 0. \tag{8}$$

By considering an arbitrary domain G_2^h consisting of elementary rectangles, which are touching each other and by summing up the type (8) equations for each rectangle, we would get the discrete analog of the conservation law for that domain.

In order to use this difference scheme for passing from layer to layer, we need to determine beforehand the values $\rho_{m+0.5}^{n+0.5}$ by taking use of the discontinuity decadence law [4].

3. The Solution of the Problem for the Three Dimensional Space. Now consider the problem (1), (2). Let us construct the following lattice in the domain under discussion:

$$G_3^n = G_3^{h,1}(\theta_m, R_k, z_n) \cup G_3^{h,2}(\theta_{m+0.5}, R_k, z_{n+0.5}) \cup G_3^{h,3}(\theta_{m+0.5}, R_k, z_n).$$

Let us define there elementary volume.

In Fig. 2 by points are designated the nodes of the lattice $G_3^{h,1}$, by crosses are designated the nodes of the lattice $G_3^{h,2}$ and by rings are designated the nodes of the lattice $G_3^{h,3}$. To solve the problem (1), (2) by the difference scheme method, let us take in the θ-axis N points, in R-axis N_1 points and in z-axis N_2 points. Set also $B = 1$ and $N_0 = 1$.

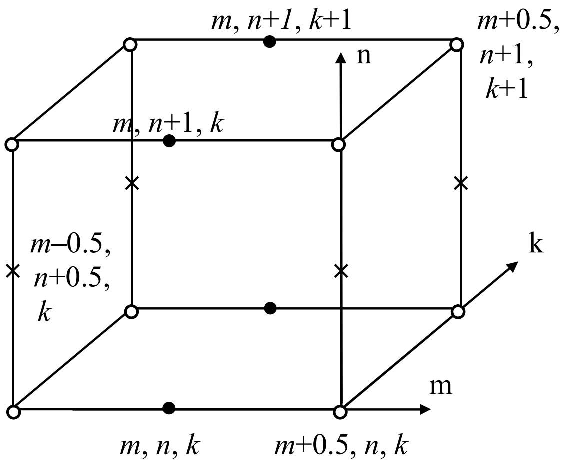

Fig. 2. The elementary lattice G_3^n .

Let us define the lattice function $\rho_{m,k}^n$. The initial conditions look like

$$\rho_{m,k}^n = F_{m,k},$$

$$h_\theta = \frac{2\pi}{N},\ \ R_k = kh_R,\ \ h_R = \frac{R_0}{N_1},$$

$$z_n = h_z n,\ \ h_z = \frac{Z}{N_2}.$$

Let us replace the Eq. (1) by the following difference equation:

$$\begin{gathered}\frac{\left(\rho_{m+0.5,k}^{n+0.5}\right)^2 - 2\left(\rho_{m-0.5,k}^{n+0.5}\right)^2 + \left(\rho_{m-1.5,k}^{n+0.5}\right)^2}{h_\theta^2} - \\ -\frac{1}{h_z}\left(\frac{\rho_{m,k}^{n+1} - \rho_{m-1,k}^{n+1}}{h_\theta} - \frac{\rho_{m,k}^{n} - \rho_{m-1,k}^{n+1}}{h_\theta}\right) + \\ +\frac{1}{h_R}\left(R_{R+0.5}\frac{\rho_{m,k+1}^{n+1} - \rho_{m,k}^{n+1}}{h_R} - R_{k-0.5}\frac{\rho_{m,k}^{n+1} - \rho_{m,k-1}^{n+1}}{h_R}\right) = 0,\end{gathered} \tag{9}$$

where $\rho_{m,k}^0 = F_{m,k}$, $\rho_{m,N_1}^{n+1} = 0$.

As in the bivariate case the values of $\rho^{n+0.5}$ are determined by the discontinuity decadence law.

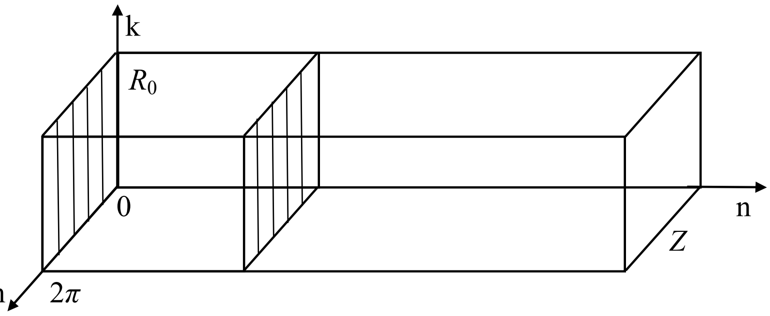

Fig. 3. Three-dimensional domain, in which we seek for a solution of equation.
The initial and n-th layers are shaded.

Let us denote

$$\Phi^n_{m,k} = \frac{\rho^n_{m,k} - \rho^n_{m-1,k}}{h_\theta} + h_z \frac{\left(\rho^{n+0.5}_{m+0.5,k}\right)^2 - 2\left(\rho^{n+0.5}_{m-0.5,k}\right)^2 + \left(\rho^{n+0.5}_{m-1.5,k}\right)^2}{h^2_\theta},$$

$$m = 0,1,\ldots,N-1;\ k = 0,1,\ldots,N_1 - 1;\ n = 0,1,\ldots,N_2 - 1.$$

Then the system (9) will take the form:

$$-\left(\frac{\rho^{n+1}_{m,k} - \rho^{n+1}_{m-1,k}}{h_\theta}\right) +$$
$$+h_z\left(R_{k+0.5}\frac{\rho^{n+1}_{m,k+1} - \rho^{n+1}_{m,k}}{h_R} - R_{k-0.5}\frac{\rho^{n+1}_{m,k} - \rho^{n+1}_{m,k-1}}{h_R}\right) = -\Phi^n_{m,k}. \quad (10)$$

Since there are 3 unknowns in each equation of the system (10), therefore, the matrix of the system is three-diagonal. Accordingly, for this special system the Gauss method for three-diagonal matrix is used [6].

To solve the system (10) with the mentioned method, we bring first the equations to the following form:

$$\frac{h_z}{h^2_R}R_{k-0.5}\rho^{n+1}_{m,k-1} - \frac{\rho^{n+1}_{m,k}}{h_\theta} + \frac{\rho^{n+1}_{m-1,k}}{h_\theta} - \frac{h_z}{h^2_R}R_{k+0.5}\rho^{n+1}_{m,k} -$$
$$-\frac{h_z}{h^2_R}R_{k-0.5}\rho^{n+1}_{m,k} + \frac{h_z}{h^2_R}R_{k+0.5}\rho^{n+1}_{m,k+1} = -\Phi^n_{m,k}.$$

From here we get

$$\frac{h_z}{h^2_R}R_{k-0.5}\rho^{n+1}_{m,k-1} - \left(\frac{1}{h_\theta} + \frac{h_z}{h^2_R}R_{k+0.5} + \frac{h_z}{h^2_R}R_{k-0.5}\right)\rho^{n+1}_{m,k} +$$
$$+\frac{1}{h_\theta}\rho^{n+1}_{m-1,k} + \frac{h_z}{h^2_R}R_{k+0.5}\rho^{n+1}_{m,k+1} = -\Phi^n_{m,k}. \quad (11)$$

By assuming that we have the vlaues $\rho^n_{m,k}$ in the n-th layer, we get first the intermediate values $\rho^{n+0.5}_{m+0.5,k}$ by using the discontinuity decadence law.

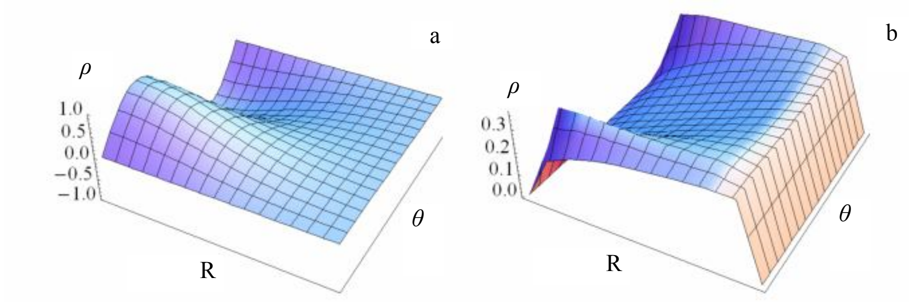

Fig. 4. Grapf of function ρ: a) initial wave; b) obtained wave on $z=1$;
$0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant\ R \leqslant 2$.

Then on each of the k-th line of the $(n+1)$-th layer we are applying the Gauss method for three-diagonal matrices. Numerical calculations were conducted for $R_0=2$ and $z \in [0;1]$. We take in the θ, R and z axis $N=20$, $N_1=10$ and $N_2=20$ points respectively. The obtained results we present in form of three-dimensional graph (Fig. 4).

As it follows from the results, that initial wave deformes, which is typical for sound waves.

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