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M a t h e m a t i c s

ON A HILBERT PROBLEM IN THE HALF-PLANE IN THE CLASS OF CONTINUOUS FUNCTIONS

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We study the Hilbert boundary value problem in the half-plane, when the boundary function is continuous on the real axis. It was proved that this problem is Noetherian and the solutions of the corresponding homogeneous problem are determined in explicit form.

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Introduction. Let $\Pi^{\pm} = \{x+iy : y \gtrless 0\}$ be the upper and lower half-planes respectively, and A is the class of analytic in $\Pi^{+} \bigcup \Pi^{-}$ functions Φ satisfying the condition

$$|\Phi(z)| \leq Bz^{m}, \;\; |Imz| \geq y > 0. \tag{1}$$

Here m is any natural number and B is a constant, in general dependent of y. We say that the function f is continuous on the real axis or $f \in C(\overline{\mathbb{R}})$, if f is continuous on $(-\infty,+\infty)$, the limits $f(\pm\infty)$ exist and $f(+\infty) = f(-\infty) = f(\infty)$.

Hilbert boundary value problem is considered. Analytic function Φ from the class A is determined, which satisfies the following boundary condition

$$\lim_{y\to+0} \|\Phi^{+}(x+iy) - a(x)\Phi^{-}(x-iy) - f(x)\|_C = 0, \tag{2}$$

where $\Phi^{\pm}$ are the restrictions of the function Φ on $\Pi^{\pm}$ respectively, where f is a given function from the class $C(\overline{\mathbb{R}})$. We suppose that the function $a(x) \neq 0$ for $x \in \mathbb{R}$, $a \in C^{\alpha}(\overline{\mathbb{R}})$, that is a satisfies the Holder condition for any finite point $x \in \mathbb{R}$, the limits $a(\pm\infty)$ exist and we have the conditions:

$$a(+\infty) = a(-\infty) = 1; \qquad |a(x_1) - a(x_2)| < C\left|\frac{1}{i+x_1} - \frac{1}{i+x_2}\right|^{\alpha}.$$

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The Problem (2) in the circle and for the bounded domains were investigated in the [1–4]. The Hilbert problem in L^1 space was considered in [5, 6].

Further, if $\Phi(z) \in A$, we denote $\Phi^{\pm}$ the restrictions of the function Φ in the half-planes $\Pi^{\pm}$ respectively and if $\Phi^{\pm}$ are analytic in $\Pi^{\pm}$, then we denote

$$\Phi(z) = \begin{cases} \Phi^+(z), & z \in \Pi^+, \\ \Phi^-(z), & z \in \Pi^-. \end{cases}$$

Let μ be an index of the function $a : \mu = inda(x),\ x \in \mathbb{R}$. Then we can represent the coefficient a in the form $a(t) = \dfrac{S^+(t)}{S^-(t)}, \quad t \in \mathbb{R}$, where

$$S^+(z) = \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln a_1(t)dt}{t-z} \right\}, \qquad z \in \Pi^+, \tag{3}$$

$$S^-(z) = \left(\frac{z+i}{z-i}\right)^{\mu} \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln a_1(t)dt}{t-z} \right\}, \qquad z \in \Pi^-, \tag{4}$$

and $a_1(t) = \left(\dfrac{t+i}{t-i}\right)^{\mu} a(t)$. If $f \in C(\overline{\mathbb{R}})$, then we denote

$$K(f,z) = \frac{(z+i)S(z)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{(t+i)S^+(t)} \cdot \frac{dt}{t-z}. \tag{5}$$

Auxiliary Results.

L e m m a 1 . For $y > 0$ we have $|S^+(x+iy) - a(x)S^-(x-iy)| < A\dfrac{y^{\alpha}}{|x+i|^{2\alpha}}$, where $A > 0$ is a constant.

P r o o f . Using (3) and (4), we get

$$|S^+(x+iy) - a(x)S^-(x-iy)| \le$$
$$\le A_y|S^+(x+iy)| \left| 1 - \exp\left(-\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y \ln a_1(t)dt}{(t-x)^2+y^2} + \ln a_1(x) \right) \right|.$$

The functions $S^+(z)$ and $(S^+(z))^{-1}$ are bounded in Π^+, and if $|z| \le M$, then $|1-e^z| < A|z|$ (A may depend on M). Therefore, we get

$$|x+i|^{2\alpha}|S^+(x+iy) - a(x)S^-(x-iy)| \le C \int_{-\infty}^{+\infty} \frac{y|x+i|^{2\alpha}|\ln a_1(t) - \ln a_1(x)|dt}{(t-x)^2+y^2} \le$$
$$\le C \int_{-\infty}^{+\infty} \frac{y|x+i|^{\alpha}|t-x|^{\alpha}dt}{|t+i|^{\alpha}((t-x)^2+y^2)} \le C(I_1(x,y) + I_2(x,y)),$$

where $I_1(x,y) = \displaystyle\int_{-\infty}^{+\infty} \frac{y|t-x|^{\alpha}dt}{(t-x)^2+y^2}$, $\quad I_2(x,y) = \displaystyle\int_{-\infty}^{+\infty} \frac{y|t-x|^{2\alpha}dt}{|t+i|^{\alpha}((t-x)^2+y^2)}$, then $I_1(x,y) = I_1^{'}(x,y) + I_1^{''}(x,y)$, where

$$I_1^{'}(x,y) = \int_{|t-x|<y} \frac{y|t-x|^{\alpha}dt}{(t-x)^2+y^2}, \qquad I_1^{''}(x,y) = \int_{|t-x|>y} \frac{y|t-x|^{\alpha}dt}{(t-x)^2+y^2}.$$

Taking into account the inequalities

$$|I_1^{'}(x,y)| \le \left| \int_{-y}^{+y} \frac{|y|^{1+\alpha}dt}{y^2} \right| = 2y^{\alpha}, \quad |I_1^{''}(x,y)| \le \left| \int_{|t|>y} \frac{y|t|^{\alpha}dt}{t^2} \right| = 2y^{\alpha},$$

we get $|I_1(x,y)| < Cy^{\alpha}$ and similarly $|I_2(x,y)| < Cy^{\alpha}$, where the constant C doesn't depend on x, y. $\square$

It must be mentioned that

$$\left\| \frac{S^+(x+iy)}{(x+iy+i)^k} - a(x)\frac{S^-(x-iy)}{(x-iy+i)^k} \right\| \leq \text{const}\, y, \;\; k \geq 1. \tag{6}$$

This inequality can be proved by using Lemma 1 as follows:

$$|S^+(x+iy)(x+iy+i)^k - a(x)S^-(x-iy)(x-iy+i)^k| \leq$$
$$\leq |(x+iy+i)^k||S^+(x+iy) - a(x)S^-(x-iy)| +$$
$$+|(x+iy+i)^k||a(x)S^-(x-iy)|\left|1 - \left(\frac{x-iy+i}{x+iy+i}\right)^k\right| \leq$$
$$\leq |(x+iy+i)^k|\left(|S^+(x+iy) - a(x)S^-(x-iy)| + \frac{A_1 y}{|x+i|}\right) \leq$$
$$\leq y(A|x+i|^{k-1} + y^{\alpha}|x+i|^{k-2\alpha}).$$

Then we assume $f \in C(\overline{\mathbb{R}})$ and

$$I_1(f,x,y) = \frac{S^+(x+iy)(x+iy+i)}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)(t+i)} \cdot \frac{y\,dt}{(t-x)^2+y^2}, \tag{7}$$

$$I_2(f,x,y) = \frac{S^+(x+iy)(x+iy+i) - a(x)S^-(x-iy)(x-iy+i)}{2\pi i} \times$$
$$\times \int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)(t+i)} \cdot \frac{dt}{t-x+iy}. \tag{8}$$

Lemma 2. Let $f \in C(\overline{\mathbb{R}})$. Then $\|I_1(f,x,y)\|_C \leq M\|f\|_C$ (M is a constant).

Proof. As $S^+(z)$ is bounded, we get

$$I_1(f,x,y) = \frac{S^+(x+iy)}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)(x+iy+i)}{S^+(t)(t+i)} \cdot \frac{y\,dt}{(t-x)^2+y^2} =$$
$$= \frac{S^+(x+iy)}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)} \cdot \frac{y\,dt}{(t-x)^2+y^2} +$$
$$+\frac{S^+(x+iy)}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)((x+iy+i)-(t+i))}{S^+(t)(t+i)} \cdot \frac{y\,dt}{(t-x)^2+y^2} = I_1^1(f,x,y) + I_1^2(f,x,y).$$

It is clear that $\|I_1^1(f,x,y)\|_C \leq M_1\|f\|_C$ and since $|(x+iy+i)-(t+i)| = |t-x-iy|$, we obtain $\|I_1^2(f,x,y)\|_C \leq M_2\|f\|_C \int_{-\infty}^{+\infty} \frac{y\,dt}{|t+i||t-x+iy|} \leq$
$\leq M_2\|f\|_C\left\{\int_{-\infty}^{+\infty} \frac{y\,dt}{|t+i|^2} + \int_{-\infty}^{+\infty} \frac{y\,dt}{|t-x+iy|^2}\right\} \leq M_3\|f\|_C$.

Lemma 3. Let $f \in C(\overline{\mathbb{R}})$. Then $\|I_2(f,x,y)\|_C \leq M\|f\|_C$ (M is a constant).

Proof. Taking into account the equality

$$S^+(x+iy)(x+iy+i) - a(x)S^-(x-iy)(x-iy+i) =$$
$$= (x+i)(S^+(x+iy) - a(x)S^-(x-iy)) + iy(S^+(x+iy) + a(x)S^-(x-iy))$$

and Lemma 1, we get

$$|I_2(f,x,y)| \leq \frac{y^{\alpha}}{|x+i|^{2\alpha-1}} \left| \int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)(t+i)} \cdot \frac{dt}{t-x+iy} \right| +$$
$$+y|S^+(x+iy)+a(x)S^-(x-iy)| \left| \int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)(t+i)} \cdot \frac{dt}{t-x+iy} \right|.$$

Since $|S^{\pm}(z)| <$const for $y>0$ and $\left\| \int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)(t+i)} \cdot \frac{dt}{t-x+iy} \right\|_C \leq M\|f\|_C$. □

Lemma 4. Let $f \in C(\overline{\mathbb{R}})$. Then $K(f,z)$ is a solution of (2), where $K(f,z)$ is defined by (5).

Theorem 1. Let $f \in C(\overline{\mathbb{R}})$ and $\Phi(z) \in A$ is the solution of (2). Then:

a) in the case $\mu \geq -1$ the function $\Phi(z)$ can be represented in the form

$$\Phi(z) = \frac{(z+i)S(z)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)(t+i)} \cdot \frac{dt}{t-z} + (z+i)S(z)G\left(\frac{1}{z+i}\right),\ z \in \Pi^{\pm}, \quad (9)$$

where G is a polynomial of order μ $(\mu > 0)$, and $G(\omega) \equiv 0$ if $\mu = 0$ or $\mu = -1$;

b) in the case $\mu < -1$ $\Phi(z)$ is represented in the form (9), where $G \equiv 0$ and the function f satisfies the orthogonality conditions

$$\int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)} \cdot \frac{dt}{(t+i)^{k+1}} = 0, \qquad k = 1,2,...,-\mu-1.$$

Proof. a) If $\mu > 0$, then we will denote

$$\Phi^+(x+iy) - a(x)\Phi^-(x-iy) = f_y(x). \quad (10)$$

Multiplying (10) by $(x+i)^{-1}$ and denoting

$\Phi_y^+(z) = \frac{\Phi^+(z+iy)}{S^+(z)(z+i)},\ z \in \Pi^+,\ \Phi_y^-(z) = \frac{\Phi^-(z-iy)}{S^-(z)(z+i)},\ z \in \Pi^-,$ we get

$$\Phi_y^+(x) - \Phi_y^-(x) = \frac{f_y(x)}{S^+(x)(x+i)}.$$

In the case $\mu \geq -1$, the function $\Phi_y^-(z)$ has a pole of order $\mu - 1$ at the point $z = -i$. This equality can be represented in the form

$$\Phi_y^+(x) - \tilde{\Phi}_y^-(x) = \frac{f_y(x)}{S^+(x)(x+i)} + G_y(x), \quad (11)$$

where $G_y(z)$ is the principal part of Loran's series for $\Phi_y^-(z)$ at the point $z=-i$: $G_y(z) = \frac{A_1(y)}{z+i} + ... + \frac{A_{k-1}(y)}{(z+i)^{k-1}}$ and $\tilde{\Phi}_y^-(z) = \Phi_y^-(z) - G_y(z)$. The solution of (11) can be written in the form

$$\Phi_y^{\pm}(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f_y(t)}{S^+(t)(t+i)} \cdot \frac{dt}{t-z} + P_y(z) + G_y(z). \quad (12)$$

We have to prove that an unknown polynomial $P_y(z) \equiv 0$.

$$\Phi^+(x+iy) - a(x)\Phi^-(x-iy) = I_1(f,x,y) - I_2(f,x,y) +$$
$$+(x+iy+i)S^+(x+iy)G\left(\frac{1}{x+iy+i}\right) - (x-iy+i)S^+(x-iy)G\left(\frac{1}{x-iy+i}\right) +$$
$$+(x+iy+i)S^+(x+iy)P_y(x+iy) - (x-iy+i)S^+(x-iy)P_y(x-iy) = f_y(x).$$

Taking into account Lemmas 1–3 and (6), we get $P_y(z) \equiv 0$ for $f_y \in C(\overline{\mathbb{R}})$.

Taking the limit of (12) as $y \to +0$, we get

$$\Phi^{\pm}(z) = \frac{(z+i)S^{\pm}(z)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{S^{+}(t)(t+i)} \cdot \frac{dt}{t-z} + (z+i)S^{\pm}(z)[P(z)+G(z)],$$

where $G(z)$ is the principal part of Loran's series for $\Phi^{-}(z)(S^{-}(z)(z+i))^{-1}\Big|_{z=-i}$.

b) In the case $\mu < -1$ the function $\Phi_y^{-}(z)$ is holomorphic in Π^{-}, therefore, the principal part of it's Loran's expansion is zero, i.e. $G(z) \equiv 0$. On the other hand, $\Phi^{-}(z)(S^{-}(z)(z+i))^{-1}$ has zero of order $|\mu-1|$ at the point $z=-i$. Consequently $f(x)$ satisfies the conditions $\displaystyle\int_{-\infty}^{+\infty} \frac{f(t)}{S^{+}(t)} \cdot \frac{dt}{(t+i)^{k+1}} = 0, \quad k = 1,2,...,-\mu-1.$ $\square$

T h e o r e m 2 . Let $f \in C(\overline{\mathbb{R}})$. Then

a) if $\mu \geq -1$, then $\Phi(z)$ the general solution of (2) is represented in the form

$$\Phi(z) = \frac{(z+i)S(z)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{S^{+}(t)(t+i)} \cdot \frac{dt}{t-z} + (z+i)S(z)G\left(\frac{1}{z+i}\right), \quad z \in \Pi^{\pm}, \tag{13}$$

where G is a polynomial of order μ if $\mu > 0$, and $G(\omega) \equiv 0$ if $\mu = 0$ or $\mu = -1$;

b) otherwise (2) is solvable if and only if the function f satisfies the conditions

$$\int_{-\infty}^{+\infty} \frac{f(t)}{S^{+}(t)} \frac{dt}{(t+i)^{k+1}} = 0, \qquad k = 1,2,...,-\mu-1.$$

The general solution can be represented by (13), where $G(z) \equiv 0$.

P r o o f . Let $f_n(x) \in C^{\alpha}(-\infty,+\infty)$ be a sequence of finite functions such that $\lim\limits_{n\to\infty} \|f_n(x) - f(x)\|_C = 0$. For arbitrary n we denote

$$\Phi_n(z) = \frac{(z+i)S(z)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f_n(t)}{S^{+}(t)(t+i)} \cdot \frac{dt}{t-z} + (z+i)S(z)G\left(\frac{1}{z+i}\right), \quad z \in \Pi^{\pm}.$$

We will prove that

$$\lim_{y\to+0} \|\Phi_n^{+}(x+iy) - a(x)\Phi_n^{-}(x-iy) - f_n(x)\|_C = 0. \tag{14}$$

From the formula of Sokhotski–Plemelj we get

$$\Phi_n^{+}(x+iy) - a(x)\Phi_n^{-}(x-iy) = f_n(x), \qquad x \in (-A,A). \tag{15}$$

If $|x| > A$, using the representation

$$\begin{aligned} &\Phi_n^{+}(x+iy) - a(x)\Phi_n^{-}(x-iy) = \\ &= \frac{S^{+}(x+iy)(x+iy+i)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f_n(t)}{S^{+}(t)(t+i)} \cdot \frac{y\,dt}{(t-x)^2+y^2} + \\ &+ \frac{S^{+}(x+iy)(x+iy+i) - a(x)S^{-}(x-iy)(x-iy+i)}{2\pi i} \times \\ &\times \int_{-\infty}^{+\infty} \frac{f(t)}{S^{+}(t)(t+i)} \cdot \frac{dt}{t-x+iy} + J(x,y), \end{aligned}$$

where $J(x,y) = \displaystyle\sum_{k=1}^{\mu} c_k \left[\frac{S^{+}(x+iy)}{(x+iy+i)^k} - \frac{S^{-}(x-iy)}{(x-iy+i)^k}\right]$, we get

$$\max_{|z|>A} \left|\Phi_n^{+}(x+iy) - a(x)\Phi_n^{-}(x-iy) - f_n(x)\right| \leq J_1(x,y) + J_2(x,y) + J_3(x,y),$$

where

$$J_1(x,y)=\max_{|z|>A}\left|\frac{S^+(x+iy)(x+iy+i)}{2\pi i}\int_{-\infty}^{+\infty}\frac{f_n(t)}{S^+(t)(t+i)}\cdot\frac{ydt}{(t-x)^2+y^2}-f_n(x)\right|,$$

$$J_2(x,y)=\max_{|z|>A}\left|\frac{S^+(x+iy)(x+iy+i)-a(x)S^-(x-iy)(x-iy+i)}{2\pi i}\times\right.$$

$$\left.\times\int_{-\infty}^{+\infty}\frac{f(t)}{S^+(t)(t+i)}\cdot\frac{dt}{t-x+iy}\right|,$$

$$J_3(x,y)=\max_{|z|>A}\left|(x+iy+i)S^+(x+iy)G\Big(\frac{1}{x+iy+i}\Big)-\right.$$

$$\left.-(x-iy+i)S^+(x-iy)G\Big(\frac{1}{x-iy+i}\Big)\right|.$$

Taking into account the relation, $f_n(x)=0$, for $|x|>A$, we get the estimation

$$J_1(x,y)\le C\max_{|z|>A}\left\{y|x+iy+i|\int_{-\infty}^{+\infty}\left|\frac{f_n(t)}{S^+(t)(t+i)}\right|\frac{ydt}{(t-x)^2+y^2}\right\}\le$$

$$\le C\|f_n\|_C\max_{|z|>A}\left\{y|x+iy+i|\int_{-\infty}^{+\infty}\frac{dt}{(t-x)^2}\right\}=C_1\|f_n\|_C\max_{|z|>A}\left\{\frac{y|x+iy+i|}{|x^2-A^2|}\right\},$$

which means that $J_1(x,y)$ tends to zero as $y\to+0$.
Using the scheme of the Lemma 3 proof, we get that $J_2(x,y)$ vanishes when $y\to+0$. Thus, taking into account (6) for $J_3(x,y)$ and equality (15), we get (14).

Using Lemmas 2, 3 and (14), we conclude

$$\|\Phi^+(x+iy)-a(x)\Phi^-(x-iy)-f(x)\|_C\le\|\Phi_n^+(x+iy)-a(x)\Phi_n^-(x-iy)-f_n(x)\|_C+$$
$$+\|f_n(x)-f(x)\|_C+\|[\Phi_n^+(x+iy)-\Phi^+(x+iy)]-a(x)[\Phi_n^-(x-iy)]-\Phi^-(x-iy)\|_C\le$$
$$\le\|\Phi_n^+(x+iy)-a(x)\Phi_n^-(x-iy)-f_n(x)\|_C+2\|f_n(x)-f(x)\|_C.$$

Hence, we get $\lim\limits_{y\to+0}\|\Phi^+(x+iy)-a(x)\Phi^-(x-iy)-f(x)\|_C=0.$

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