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Mathematics

DUALITY IN SPACES OF FUNCTIONS PLURIHARMONIC IN THE UNIT BALL IN $\mathbb{C}^n$

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Banach spaces $h_\infty(\Phi)$, $h_0(\Phi)$ and $h^1(\eta)$ of functions, pluriharmonic in the unit ball in $\mathbb{C}^n$, depending on weight function Φ and weighting measure η are introduced. The question we consider is: for given Φ we find a finite positive Borel measure η on $[0,1)$ such that $h^1(\eta)^* \sim h_\infty(\Phi)$ and $h_0(\Phi)^* \sim h^1(\eta)$.

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Introduction. A positive, continuous, decreasing function Φ on $[0,1)$ is called a weight function, if

$$\lim_{r\to 1}\Phi(r)=0.$$

A positive finite Borel measure η on $[0,1)$ is called a weighting measure, if it is not supported in any subinterval $[0,\rho)$, $0<\rho<1$.

In the present work we consider the duality problem in the case of pluriharmonic functions in the unit ball of $\mathbb{C}^n$, $n\geq 2$.

The following notation is used: $B=\{z\in\mathbb{C}^n;|z|<1\}$ is the open unit ball in $\mathbb{C}^n$; $S=\{z\in\mathbb{C}^n;|z|=1\}$ is the unit sphere in $\mathbb{C}^n$; $h(B)$ is the vector space of complex-valued functions, pluriharmonic in B, with the usual pointwise addition and scalar multiplication; σ stands for the Lebesgue measure of the area element on S, normalized by the condition $\sigma(S)=1$.

Let $\Phi(r)$ be a weight function and η be a weighting measure. We extend Φ to the whole B by $\Phi(z)=\Phi(|z|)$.

For $u\in h(B)$ we define

$$\|u\|_\Phi=\sup\{|u(z)|\Phi(z);z\in B\}=\sup\{M_\infty(u,r)\Phi(r);r<1\},$$

$$\|u\|_\eta=\int_S\int_0^1|u(r\xi)|\,d\eta(r)d\sigma(\xi)=\int_0^1 M_1(u,r)\,d\eta(r),$$

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where $M_\infty(u,r)=\sup\{|u(z)|;\,|z|=r\}$, $M_1(u,r)=\int_S|u(r\xi)|\,d\sigma(\xi)$.

We define following spaces of pluriharmonic functions:

$$h_\infty(\Phi)=\{u\in h(B);\|u\|_\Phi<\infty\},$$

$$h_0(\Phi)=\{u\in h(B);\lim_{r\to 1}M_\infty(u,r)\Phi(r)=0\},$$

$$h^1(\eta)=\{u\in h(B);\|u\|_\eta<\infty\}.$$

These are all normed linear spaces. Obviously, $h_0(\Phi)\subset h_\infty(\Phi)$, so we can use the norm $\|u\|_\Phi$ on $h_0(\Phi)$.

For the pluriharmonic functions the following problem is solved: for given weight function Φ we find a finite positive Borel measure η on $[0,1)$ such that $h^1(\eta)^*\sim h_\infty(\Phi)$ and $h_0(\Phi)^*\sim h^1(\eta)$. For analytic and harmonic functions in the unit disk on the complex plane, Shields and Williams [1, 2] posed and solved the mentioned problem for the first time. Recently in [5], the problem was solved for the harmonic functions in the unit ball of $\mathbb{R}^n$.

Preliminaries. In the following two propositions we establish some basic facts about these spaces.

Proposition 1. Let h denote any of the spaces $h_\infty(\Phi)$,$h_0(\Phi)$ or $h^1(\eta)$. Then:

(i) if b is a bounded subset of h, then the functions in b are uniformly bounded on each compact subset of B;

(ii) if u_n is the Cauchy sequence in h, then it converges uniformly on each compact subset of B;

(iii) point evaluation at any point of B is a bounded linear functional on h;

(iv) h is a Banach space;

(v) $h_0(\Phi)$ is a closed subspace of $h_\infty(\Phi)$.

Proof. In [3] (Proposition 2) in terms of our notation the following inequality was obtained for $u\in h^1(\eta)$:

$$|u(z)|\le\frac{2^{2n}}{(1-|z|)^{2n-1}}\left(\int_{(1+|z|)/2}^{1}|d\eta(r)|\right)^{-1}\|u\|_\eta,\qquad z\in B.$$

This gives us (i) and (iii) for $u\in h^1(\eta)$. For $h_\infty(\Phi)$ and $h_0(\Phi)$ these statements are obvious. It is easy to see also that (ii) follows from (i).

Now we prove (iv). It is only necessary to establish the completeness, and this is easy for $h_\infty(\Phi)$ and $h_0(\Phi)$. Let $u_j\in h_\infty(\Phi)$. The function $\Phi(z)$ is away from zero on a compact subset K, hence,

$$|u_j(z)-u_k(z)|\le C\varphi(z)|u_j(z)-u_k(z)|=C\|u_j-u_k\|_\Phi,\qquad z\in K.$$

Since the sequence u_j is fundamental in $h_\infty(\Phi)$, u_j converges uniformly to some function u on the compact subsets of B, which is pluriharmonic in B. Obviously u_j converges to u also in $h_\infty(\Phi)$.

Now let the sequence u_j be fundamental in $h^1(\eta)$, and let K be a compact subset of B. It follows from (ii) that there is a constant $C=C(K)$ such that

$\max_{z\in K}|u(z)| \le C\|u\|_\eta$ for all $u \in h^1(\eta)$. Therefore, $|u_j(z) - u_k(z)| \le C\|u_j - u_k\|_\eta$ for any $z \in K$ and j,k. Since u_j is fundamental in $h^1(\eta)$, the sequence u_j converges uniformly to some pluriharmonic in B function u on compact the subsets of B. Again, since the sequence u_j is fundamental, we have

$$\int_S\int_0^1 \mid u_j(r\xi) \mid d\eta(r)d\sigma(\xi) = \|u_j\|_\eta \le \|u_j - u_k\|_\eta + \|u_k\|_\eta \le C.$$

By Fatou's lemma, $\|u\|_\eta = \int_S\int_0^1 \mid u(r\xi) \mid d\eta(r)d\sigma(\xi) \le C, \quad$ i.e. $\quad u \in h^1(\eta)$.

(v) follows from (iv). □

For $u \in h(B)$ define $u_\rho(z) = u(\rho z)$, $0 \le \rho \le 1$.

P r o p o s i t i o n 2.

(i) For $u \in h^1(\eta)$ or $u \in h_0(\Phi)$ $u_\rho \to u$ in norm as $\rho \to 1$.

(ii) For $u \in h_\infty(\Phi)$, $\|u_\rho\|_\Phi \le \|u\|_\Phi$ and $u_\rho \to u$ pointwise in B.

(iii) The pluriharmonic polynomials are dense in $h^1(\eta)$ and in $h_0(\Phi)$.

(iv) Each $u \in h_\infty(\Phi)$ is pointwise limit of some sequence of polynomials bounded in norm.

P r o o f . (i) This is obvious for $h_0(\Phi)$. For $u \in h^1(\eta)$ and $\varepsilon > 0$ choose $\delta < 1$, so that

$$\int_\delta^1 M_1(u,r)\,d\eta(r) < \varepsilon.$$

Since $M_1(u_\rho, r) \le M_1(u,r)$, we have

$$\int_\delta^1 M_1(u_\rho,r)\,d\eta(r) < \varepsilon, \qquad 0 \le \rho \le 1.$$

Choose ρ so that $|u - u_\rho| < \varepsilon$ on $|z| \le \delta$. Then

$$\|u_\rho - u\|_\eta = \int_0^1\int_S |u(\rho r\xi) - u(r\xi)|\,d\sigma(\xi)d\eta(r) \le$$

$$\le \int_0^\delta\int_S |u(\rho r\xi) - u(r\xi)|\,d\sigma(\xi)d\eta(r) + \int_\delta^1\int_S (|u(\rho r\xi)| + |u(r\xi)|)\,d\sigma(\xi)d\eta(r)$$

$$\le \varepsilon \int_{|z|\le\delta} d\eta(r)d\sigma(\xi) + 2\varepsilon.$$

Part (ii) is obviously obtained from the maximum principle and continuity.

It is well known that any function pluriharmonic in a neighborhood of $\bar{B}$ can be approximated uniformly on $\bar{B}$ by pluriharmonic polynomials. Using this fact we deduce (iii) and (iv). □

Duality, Projections, Reproducing Kernels. Given a weight function Φ, we wish to find a weighting measure η, such that the following duality relations hold:

$$h^1(\eta)^* \sim h_\infty(\Phi), \qquad h_0(\Phi)^* \sim h^1(\eta).$$

That is, if $u \in h_\infty(\Phi)$, and we define $l_u(v) = \langle u, v\rangle$ for $v \in h^1(\eta)$, then $l_u \in h^1(\eta)^*$ and $\|l_u\| \le \|u\|_\Phi$.

Conversely, given $l \in h^1(\eta)^*$, then there is a unique $u \in h_\infty(\Phi)$ such that $l = l_u$ and $\|u\|_\Phi \le C\|l\|$. Similar remarks apply to the other duality relation.

Let Φ be any weight function and η any weighting measure. We introduce the measure $d\mu = \Phi d\eta$ and the measure $\mu^{'}$, defined by

$$\int_0^1 f(r)\,d\mu^{'}(r) = \int_0^1 f(r^2)\,d\mu(r), \qquad f \in C[0,1].$$

Proposition 3. Let

$$\begin{cases} t_n = \displaystyle\int_0^1 r^n\,d\mu^{'}(r) = \int_0^1 r^{2n}\,d\mu(r), \qquad n = 0,1,2,\dots \\ k_w(z) = \displaystyle\sum_{|p|=0}^{\infty} t_{|p|}^{-1} c(n,|p|)\langle w,z\rangle^{|p|} + \sum_{|p|=1}^{\infty} t_{|p|}^{-1} c(n,|p|)\langle z,w\rangle^{|p|}, \end{cases} \tag{1}$$

where $\langle z,w\rangle = \sum\limits_{k=1}^{n} z_k \bar{w}_k, \qquad z,w \in \mathbb{C}^n,$

$$c(n,|p|) = \frac{(n-1+|p|)!}{(n-1)!|p|!}$$

and p is a n-index: $p = (p_1, p_2, \dots, p_n), \quad p! = \prod_{k=1}^{n} p_k!, \quad |p| = \sum_{k=1}^{n} p_k$.
Define

$$\langle u,v\rangle = \int_S \int_0^1 u(r\xi)v(r\bar{\xi})\Phi(r)\,d\eta(r)d\sigma(\xi), \qquad u \in h_\infty(\Phi), v \in h^1(\eta). \tag{2}$$

Then the function k_w is pluriharmonic in $\{z; |z| < |w|^{-1}\}$, and is the reproducing kernel associated with bilinear form (2), i.e. $u(w) = \langle u, k_w\rangle$ for all $u \in h_\infty(\Phi)$ and $v(w) = \langle k_w, v\rangle$ for all $v \in h^1(\eta)$.

Proof. Since the measure $\mu^{'}$ does not vanish in any neighborhood of 1, then

$$t_k \geq \int_\rho^1 r^k\,d\mu^{'}(r) \geq \rho^k \int_\rho^1 d\mu^{'}(r), \quad k = 0,1,2,\dots,$$

implying that $t_k^{-1} = O(\rho^{-k}), k \to \infty$ for each $0 < \rho < 1$.

Then, using inequality $|\langle z,w\rangle| \leq |z||w|$ and Stirling's formula, we obtain that the series in the right hand side of (1) converges absolutely and uniformly on each compact subset, in the space $\mathbb{C}^n \times \mathbb{C}^n$. So for each fixed z, k_w is pluriharmonic in $\{z, |z| < |w|^{-1}\}$ and consequently $k_w \in h_0(\Phi) \cap h^1(\eta)$.

Consider

$$\int_S \langle w,\xi\rangle^k \xi^p\,d\sigma(\xi) = \int_S \left(\sum_{|\beta|=k} \frac{|\beta|!}{\beta!} w^\beta \bar{\xi}^\beta\right) \xi^p\,d\sigma(\xi) =$$

$$= \frac{|p|!}{p!} w^p \int_S \xi^p \bar{\xi}^p\,d\sigma(\xi) = \frac{|p|!}{p!}\frac{(n-1)!p!}{(n-1+|p|)!} w^p = w^p \frac{1}{c(n,|p|)}.$$

Now let $P_k(\xi) = \sum_{|p|=k} a_p \xi^p$ be a homogeneous polynomial of degree k and consider

$$\int_S \langle w,\xi\rangle^k P_k(\xi)\,d\sigma(\xi) = \sum_{|p|=k} a_p \int_S \langle w,\xi\rangle^k \xi^p\,d\sigma(\xi) =$$

$$= \sum_{|p|=k} a^p w^p \frac{1}{c(n,|p|)} = \frac{1}{c(n,k)} P_k(w),$$

hence

$$\int_B \langle w,z\rangle^k P_k(z)\,d\nu(z) = \int_B \langle w,r\xi\rangle^k P_k(r\xi)\,d\nu(r\xi) =$$
$$= \int_0^1 r^{2k}\,d\mu(r)\int_S \langle w,\xi\rangle^k P_k(\xi)\,d\sigma(\xi) = \frac{t_k P_k(w)}{c(n,k)}.$$

So

$$\int_B \langle w,z\rangle^k P_k(z)\,d\nu(z) = \frac{t_k P_k(w)}{c(n,k)}. \tag{3}$$

On the other hand,

$$\int_S \langle \xi,w\rangle^k \bar{\xi}^p\,d\sigma(\xi) = \int_S \left(\sum_{|\beta|=k}\frac{|\beta|!}{\beta!}\xi^\beta \bar{w}^\beta\right)\bar{\xi}^p\,d\sigma(\xi) =$$
$$= \frac{|p|!}{p!}\bar{w}^p\int_S \xi^p\bar{\xi}^p\,d\sigma(\xi) = \frac{|p|!}{p!}\frac{(n-1)!p!}{(n-1+|p|)!}\bar{w}^p = \bar{w}^p\frac{1}{c(n,|p|)}$$

and if $P_k(\bar{\xi}) = \sum_{|p|=k} a_p\bar{\xi}^p$, then we get

$$\int_S \langle \xi,w\rangle^k P_k(\bar{\xi})\,d\sigma(\xi) = \frac{P_k(\bar{w})}{c(n,k)}$$

and

$$\int_S \langle z,w\rangle^k P_k(\bar{z})\,d\sigma(z) = \frac{t_k P_k(\bar{w})}{c(n,k)}. \tag{4}$$

Now consider

$$u(z) = \sum_{k=0}^{l}\sum_{|m|=k} c_{|m|}z^m + \sum_{k=1}^{l}\sum_{|m|=k} b_{|m|}\bar{z}^m = \sum_{k=0}^{l} P_k(z) + \sum_{k=1}^{q} P_k(\bar{z}).$$

From (3) and (4) it follows that

$$\langle u,k_w\rangle = \int_S\int_0^1 \left(\sum_{k=0}^{l} P_k(z) + \sum_{k=1}^{q} P_k(\bar{z})\right)\times$$
$$\times\left(\sum_{|p|=0}^{\infty} t_{|p|}^{-1}c(n,|p|)\langle w,z\rangle^{|p|} + \sum_{|p|=1}^{\infty} t_{|p|}^{-1}c(n,|p|)\langle z,w\rangle^{|p|}\right)d\mu(r)d\sigma(\xi) = u(w)$$

for all the polynomials u.

Thus, applying Propositions 1,(iii) and 2 we get that the equality $\langle k_w,u\rangle =$ $= \langle u,k_w\rangle$ holds for any polinomial and therefor for any $v\in h^1(\eta)$. Finally, for a fixed w the linear functional $l(u) = \langle u,k_w\rangle$ is w^* continuous on h_∞. So $l(u) = u(w)$ when u is a polynomial. According to Proposition 2,(iv), each $u\in h_\infty$ is the (norm) bounded pointwise limit of a sequence of pluriharmonic polynomials. And this completes the proof. □

P r o p o s i t i o n 4. The subspaces of $h_o(\Phi)$ and $h^1(\eta)$ spanned by $\{k_w, w \in B\}$ are dense.

P r o o f . Consider $h^1(\eta)$, the proof for $h_0(\Phi)$ is similar. By the Hahn-Banach theorem, it suffices to show that if $l \in h^1(\eta)^*$ and $l(k_w) = 0$ for all $w \in B$, then l is zero functional. Let $\hat{l}(p)$ denote the value of l at $(r\xi)^p$. Since the sequences $\{z^p\}_0^\infty$ and $\{\bar{z}^p\}_0^\infty$ are bounded in the norm of $h^1(\eta)$, the sequence $\{\hat{l}(p)\}_{-\infty}^\infty$ is bounded too. Since $k_w(z)$ is pluriharmonic for $|z| < |w|^{-1}$, the series for k_w converges uniformly in B and hence in the norm of $h^1(\eta)$. Thus

$$l(k_w) = \sum_{|p|=0}^{\infty} t_{|p|}^{-1} c(n,|p|) \sum_{|p|=k} \frac{k!}{p!} \bar{w}^p \hat{l}(p) + \sum_{|p|=1}^{\infty} t_{|p|}^{-1} c(n,|p|) \sum_{|p|=k} \frac{k!}{p!} \bar{w}^p \hat{l}(-p).$$

From the estimate of $t_{|p|}^{-1}$ given in the proof of Proposition 3, this series converges uniformly in w on the compact subsets of B. Hence $l(k_w)$ is a pluriharmonic function of w in B. Since $l(k_w) = 0$, we have $\hat{l}(p) \equiv 0$. Thus, l annihilates all pluriharmonic polynomials, and so l annihilates $h^1(\eta)$ by Proposition 2,(iii). □

For a weighting measure η let $L^1(d\eta d\sigma)$ and $L_\infty(d\eta d\sigma)$ denote respectively the Banach spaces of complex-valued integrable and essentially bounded measurable functions associated with the measure $d\eta d\sigma$ on B. Denote the norms of these spaces by $\|\cdot\|_\eta$ and $\|\cdot\|_\infty$ respectively. Let $C_0(B)$ be the Banach space of complex-valued continuous functions on $\bar{B}$ that vanish on S, with the supremum norm. It is well known that the dual of $C_0(B)$ is $M(B)$, the space of finite complex Borel measures on B with the total variation norm. We identify $L^1(d\eta d\sigma)$ with the absolutely continuous measures with respect to $d\eta d\sigma$.

T h e o r e m . Let Φ be a weight function, η be a weighting measure and k_w be the corresponding reproducing kernel. Consider the integral linear operators

$$(Tf)(w) = \int_S \int_0^1 k_w(r\bar{\xi}) f(r\xi) d\eta(r) d\sigma(\xi), \quad f \in L_\infty(d\eta\, d\sigma),$$

$$(S\nu)(w) = \int_B k_w(\bar{z})\, \Phi(z)\, d\nu(z), \quad \nu \in M(B).$$

The following conditions are equivalent:

(i) $\|k_w\|_\eta \leq C/\Phi(w), \quad w \in B$.

(ii) T is a bounded operator from $L_\infty(d\eta\, d\sigma)$ into $h_\infty(\Phi)$.

(iii) S is a bounded operator from $M(B)$ into $h^1(\eta)$.

(iv) $h^1(\eta)^* \sim h_\infty(\Phi)$.

(v) $h_0(\Phi)^* \sim h^1(\eta)$.

P r o o f . The implication (i) $\Rightarrow$ (ii) is immediate from the definition of T. Indeed,

$$|(Tf)(w)| \leq \int_S \int_0^1 |k_w(r\bar{\xi})| \cdot |f(r\xi)| d\eta(r) d\sigma(\xi) \leq$$

$$\leq \|f\|_\infty \int_S \int_0^1 |k_w(r\bar{\xi})| d\eta(r) d\sigma(\xi) = \|f\|_\infty \|k\|_\eta \leq \frac{C}{\Phi(w)} \|f\|_\infty.$$

It follows from Fubini's Theorem, that (i) implies (iii)

$$\|S\nu\|_\eta = \int_0^1 \int_S |S\nu(r\xi)|\, d\eta(r) d\sigma(\xi) \le$$
$$\le \int_0^1 \int_S \int_B |k_w(\bar{z})|\, \Phi(z)\, d|\nu|(z) d\eta(r) d\sigma(\xi) \le \int_B \frac{C}{\Phi(z)} \Phi(z)\, d|\nu|(z) = C|\nu|$$

(because $\int_B |k_w(\bar{z})|\, d\eta(r) d\sigma(\xi) \le C/\Phi(z)$).

Now we show that (iii)⇒(v). Let $v \in h^1(\eta)$, denote $l_v(u) = \langle u, v\rangle$ for each $u \in h_0(\Phi)$. According to the Holder's inequality, $|l_v(u)| \le \|u\|_\Phi \|v\|_\eta$. Hence $\|l_v\| \le \|v\|_\eta$ and $l_v \in h_0(\Phi)^*$. We also have uniqueness: if $l_v(u) = 0$, $\forall u \in h_0$, then $v = 0$ and it follows from the relation $v(w) = l_v(k_w)$. It remains to prove that for each $l \in h_0(\Phi)^*$ there exists a unique $v \in h^1(\eta)$ such that $l = l_v$ (where $l_v(u) = \langle u, v\rangle$) and $\|v\|_\eta \le C\|l\|$. Let $h_0(\Phi)$ be a subspace of $C_0(B)$. Identifying $u \in h_0(\Phi)$ with $u\Phi \in C_0(B)$, it follows from the Hahn–Banach Theorem that there exists $\nu \in M(B)$ such that $\|l\| = \|\nu\|$ and $l(u) = \int u(\bar{z})\Phi(z)\, d\nu(z), u \in h_0(\Phi)$. Thus $l(k_w) = \int k_w(\bar{z})\Phi(z)\, d\nu(z) = S\nu(w)$.

Let $v = S\nu$. By (iii) $v \in h^1(\eta)$ and $\|v\| \le \|S\|\|\nu\| = \|S\|\|l\|$. Also, by Proposition 3, $l_v(k_w) = \langle k_w, v\rangle = v(w) = S\nu(w) = l(k_w)$. Thus l and l_v agree on the subspace of $h_0(\Phi)$ spanned by $k_w, w \in B$, and hence, by Proposition 4, $l = l_v$.

The proof that (ii) implies (iv) can be similarly deduced from, using the duality of $L^1(d\eta\, d\sigma)$ and $L_\infty(d\eta\, d\sigma)$ instead of $C_0(B)$ and $M(B)$.

Assuming (iv), the Hahn–Banach Theorem gives

$$\|k_w\|_\eta \le C\|l_{k_w}(u)\| = C\sup\{|\langle u, k_w\rangle|, u \in h_\infty(\Phi); \|u\|_\Phi \le 1\} =$$
$$= C\sup\{|u(w)|; u \in h_\infty(\Phi); \|u\|_\Phi \le 1\} \le \frac{C}{\Phi(w)}.$$

Similarly (v) implies (i). □

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