MOORE–PENROSE INVERSE OF BIDIAGONAL MATRICES. IV

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The present work completes a research started in the papers \cite{1–3}. Based on the results obtained in the previous papers, here we give a definitive solution to the problem of the Moore–Penrose inversion of singular upper bidiagonal matrices.


Keywords: generalized inverse, Moore–Penrose inverse, bidiagonal matrix.

Introduction. We consider a problem of the Moore–Penrose inversion of singular upper bidiagonal matrices

\[
A = \begin{bmatrix}
d_1 & b_1 \\
d_2 & b_2 & 0 \\
& \ddots & \ddots \\
0 & b_{n-1} & d_n
\end{bmatrix}
\]  \hspace{1cm} (1)

under the assumption \(b_1, b_2, \ldots, b_{n-1} \neq 0\) (note that this assumption does not restrict the generality of the problem, since if some of over-diagonal entries of the matrix \(A\) are zero, then the original problem is decomposed into several similar problems for bidiagonal matrices of lower order). In \cite{1} we obtained a solution to the problem in a special case, where \(d_1, d_2, \ldots, d_{n-1} \neq 0, d_n = 0\).

To solve the problem for any arrangement of one or more zeros on the main diagonal of the matrix \(A\), in \cite{2,3} we carried out some preliminary constructions and calculations. At first, we represented the matrix (1) in the block form

\[
A = \begin{bmatrix}
A_1 & B_1 \\
A_2 & B_2 \\
& \ddots & \ddots \\
A_{m-1} & B_{m-1} \\
A_m
\end{bmatrix}
\]  \hspace{1cm} (2)

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with diagonal blocks $A_k$, $k = 1, 2, \ldots, m$, of the size $n_k \times n_k$ and over-diagonal blocks $B_k$, $k = 1, 2, \ldots, m - 1$, of the size $n_k \times n_{k+1}$, where $n_1 + n_2 + \cdots + n_m = n$. The structure of the blocks was specified in the Introduction of [2]; ibid the types 1, 2 and 3 of the blocks $A_k$ have been identified. Note that by virtue of the partitioning rule, only the last block $A_m$ in (2) can be a block of type 3. The blocks $B_k$ are given in (5) of [2].

As has been shown in [2] (see A Way of Computing the Moore–Penrose Inversion), the matrix $A^+$ has the following block form:

$$A^+ = \begin{bmatrix}
Z_1 & & \\
H_2 & Z_2 & 0 \\
& \ddots & \ddots \ \\
0 & H_{m-1} & Z_{m-1} \\
& & H_m \end{bmatrix}; \quad (3)
$$

the blocks $Z_k$ and $H_k$ are computed by the formulae

$$Z_k = \lim_{\varepsilon \to +0} L_k(\varepsilon)^{-1}A_k^T, \quad k = 1, 2, \ldots, m; \quad (4)$$

$$H_k = \lim_{\varepsilon \to +0} L_k(\varepsilon)^{-1}B_k^T, \quad k = 2, 3, \ldots, m; \quad (5)
$$

where

$$L_1(\varepsilon) = A_1^T A_1 + \varepsilon I_1, \quad (6)$$

$$L_k(\varepsilon) = A_k^T A_k + B_k^{T}_{k-1} B_{k-1} + \varepsilon I_k, \quad k = 2, 3, \ldots, m, \quad (7)$$

and $I_k$ stands for the identity matrix of the order $n_k$.

For the purpose of simplifying the record of subsequent formulae, let us write the block $A_k$, $1 \leq k \leq m$, in the form

$$A_k = \begin{bmatrix}
d^{(k)}_1 & b^{(k)}_1 \\
d^{(k)}_2 & b^{(k)}_2 & 0 \\
& \ddots & \ddots \\
0 & d^{(k)}_{m-1} & b^{(k)}_{m-1} \\
& & & d^{(k)}_n \end{bmatrix}, \quad (8)
$$

where, according to (1),

$$d^{(k)}_i = d_{n_1+\ldots+n_{i-1}+i}, \quad i = 1, 2, \ldots, n_k, \quad (9)$$

$$b^{(k)}_i = b_{n_1+\ldots+n_{i-1}+i}, \quad i = 1, 2, \ldots, n_k - 1.$$

We introduce the following notation:

$$r^{(k)}_s = \frac{p^{(k)}_s}{d^{(k)}_s}, \quad s = 1, 2, \ldots, n_k - 1; \quad r^{(k)}_0 = r^{(k)}_{n_k} = 1. \quad (10)$$

Further, let

$$\Delta_k = b_{n_1+n_2+\ldots+n_k}, \quad k = 1, 2, \ldots, m - 1 \quad (11)$$

(see (5) in [2]).

Based on the results obtained in the previous articles [1][3], below we give a closed form expressions for the entries of the matrix $A^+$ as well as a numerical algorithm for their computation.
Computation of the Blocks $Z_k$. Let us start with the block $Z_1$. The problem of computing this block was discussed in [3] (see Block $Z_1$). If the corresponding block $A_k$ is of type 1, then the entries of the block $Z_1 = A_1^+$ are computed using the formulae (50)–(52) from [1]. If $A_1$ is a block of type 2, then $Z_1 = [0]_{1 \times 1}$ for $n_1 = 1$; for $n_1 \geq 2$ the block $Z_1$ is the lower bidiagonal matrix given in (17) of [2].

Note that if $m = 1$ (see the block representation (3) of the matrix $A$), then obviously $A^+ = Z_1$.

Let us discuss the blocks $Z_k$, $2 \leq k \leq m$. If $A_k$ is a block of type 1, the formulae for the entries of the block $Z_k$ are actually obtained in Lemma 3 of [3] (replacing $n$ with $n_k$ and taking into account notation (9),(10)). If $A_k$ is a block of type 2, the entries of the block $Z_k$ are derived in Lemma 5 of [3] (replacing $n$ with $n_k$ and using notation (9)). As has been said above, only the last block $A_m$ in (2) can be a block of type 3. In this case the entries of the block $Z_m$ are computed by the formulae derived in Lemma 1 of [3] (replacing $n$ with $n_m$, $A$ with $A_{m-1}$ and taking into account notation (9),(10)).

Thus, we arrive at the following statement.

**Theorem 1.** Let a singular upper bidiagonal matrix $A$ from (1) with non-zero over-diagonal entries is represented in the block form (2), according to the rule described in Introduction of [2]. Then the entries of diagonal blocks $Z_k = [z_{ij}^{(k)}]_{n_k \times n_k}$, $1 \leq k \leq m$, in the block representation (3) of the matrix $A^+$ are computed as follows.

I. The entries of the block $Z_1$:

1) if $A_1$ is a block of type 1, then
   \[ z_{ij}^{(1)} = (-1)^{i+j} \sum_{k=1}^{n_1-j} \left( \prod_{s=j}^{n_1-k} r_s^{(j)} \right) \left( \prod_{s=n_1-k+1}^{n_1-1} r_s^{(j)} \right) \]
   \[ \prod_{s=1}^{j-1} r_s^{(j)} \cdot d_j^{(1)} \sum_{k=1}^{n_1-j} \left( \prod_{s=j}^{n_1-k} r_s^{(j)} \right) \]
   \[ \prod_{s=n_1-k+1}^{n_1-1} r_s^{(j)} \]
2) if $A_1$ is a block of type 2, then
   for $n_1 = 1$:
   \[ Z_1 = [0]_{1 \times 1} \]
   for $n_1 \geq 2$:
   \[ z_{ii}^{(1)} = \frac{1}{b_{i-1}^{(i)}}, \quad i = 2, 3, \ldots, n_1, \]
   \[ z_{ij}^{(1)} = 0 \quad \text{in the remaining cases.} \]
II. The entries of the blocks $Z_k$, $2 \leq k \leq m$:

3) if $A_k$ is a block of type 1, then

3a) for the indeces $j = 1, 2, \ldots, n_k$ and $i = 1, 2, \ldots, j$:

$$z_{ij}^{(k)} = 0;$$

3b) for the indeces $j = 1, 2, \ldots, n_k - 1$ and $i = j + 1, j + 2, \ldots, n_k$:

$$z_{ij}^{(k)} = \frac{(-1)^{i+j+1} i-1}{d_j^{(k)}} \prod_{s=j}^{i-1} \frac{1}{r_s^{(k)}};$$

4) if $A_k$ is a block of type 2, then

for $n_k = 1$:

$$Z_k = [0]_{1 \times 1};$$

for $n_k \geq 2$:

$$z_{i-1}^{(k)} = \frac{1}{b_i^{(k)}}, \quad i = 2, 3, \ldots, n_k,$$

$$z_{ij}^{(k)} = 0 \quad \text{in the remaining cases};$$

5) if $A_m$ is a block of type 3 and $n_m = 1$, then

$$Z_m = \begin{bmatrix} d_1^{(m)} \\ d_1^{(m)^2} + \Delta_2^{m-1} \\ \vdots \\ \Delta_2^{m-1} \end{bmatrix}_{1 \times 1};$$

6) if $A_m$ is a block of type 3 and $n_m \geq 2$, then

6a) for the indeces $j = 1, 2, \ldots, n_m$ and $i = 1, 2, \ldots, j$:

$$z_{ij}^{(m)} = \frac{(-1)^{i+j} \prod_{s=i}^{j-1} \frac{1}{r_s^{(m)}} + \Delta_2^{m-1} \sum_{k=1}^{j-1} \frac{1}{b_k^{(m)}} \left( \prod_{s=1}^{k-1} r_s^{(m)} \right) \left( \prod_{s=k+1}^{j-1} r_s^{(m)} \right) \kappa_j^{(m)}}{d_m^{(m)^2} \prod_{s=1}^{n_m-1} \frac{1}{r_s^{(m)}} + \Delta_2^{m-1} \sum_{k=1}^{n_m-1} \frac{1}{b_k^{(m)}} \left( \prod_{s=1}^{k-1} r_s^{(m)} \right) \left( \prod_{s=k+1}^{n_m-1} r_s^{(m)} \right) + D^{(m)}},$$

where

$$\kappa_j^{(m)} = \frac{d_m^{(m)^2} \prod_{s=j}^{n_m-1} \frac{1}{r_s^{(m)}}}{d_m^{(m)^2} \prod_{s=j}^{n_m-1} \frac{1}{r_s^{(m)}}}, \quad D^{(m)} = \Delta_2^{m-1} \prod_{s=1}^{n_m-1} r_s^{(m)};$$

6b) for the indeces $j = 1, 2, \ldots, n_m - 1$ and $i = j + 1, j + 2, \ldots, n_m$:

$$z_{ij}^{(m)} = \frac{(-1)^{i+j+1} \prod_{s=i}^{j-1} \frac{1}{r_s^{(m)}} + d_m^{(m)^2} \sum_{k=1}^{n_m-1} \frac{1}{b_k^{(m)}} \left( \prod_{s=1}^{k-1} r_s^{(m)} \right) \left( \prod_{s=k+1}^{n_m-1} r_s^{(m)} \right) \omega_j^{(m)}}{d_m^{(m)^2} \prod_{s=1}^{n_m-1} \frac{1}{r_s^{(m)}} + \Delta_2^{m-1} \sum_{k=1}^{n_m-1} \frac{1}{b_k^{(m)}} \left( \prod_{s=1}^{k-1} r_s^{(m)} \right) \left( \prod_{s=k+1}^{n_m-1} r_s^{(m)} \right) + D^{(m)}},$$

where

$$\omega_j^{(m)} = \frac{\Delta_2^{m-1} \prod_{s=1}^{j-1} r_s^{(m)}}{d_m^{(m)^2} \prod_{s=1}^{j-1} r_s^{(m)}}, \quad D^{(m)} = \Delta_2^{m-1} \prod_{s=1}^{n_m-1} r_s^{(m)}.$$
Computation of the Blocks $H_k$. Proceed to the blocks $H_k$, $2 \leq k \leq m$, in the block representation (3) of the matrix $A^+$. If $A_k$ is a block of type 1, the entries of the corresponding block $H_k$ are computed by the formulae derived in Lemma 4 of [3] (naturally, replacing $n$ with $n_k$, $l$ with $n_k-1$, $\Delta$ with $\Delta_{k-1}$ and taking into account notation (9),(10)). Further, if $A_k$ is a block of type 2, the entries of the block $H_k$ are computed according to Lemma 6 from [3] (replacing $\Delta$ with $\Delta_{k-1}$). Finally, if $A_m$ is a block of type 3, the entries of the block $H_m$ are computed by the formulae derived in Lemma 2 of [3] (replacing $n$ with $n_m$, $l$ with $n_m-1$, $\Delta$ with $\Delta_{m-1}$ and taking into account notation (9),(10)).

As a result we get the following statement:

**Theorem 2.** Let a singular upper bidiagonal matrix $A$ from (1) with non-zero over-diagonal entries is represented in the block form (2), according to the rule described in Introduction of [2]. Then the entries of under-diagonal blocks $H_k = [h_{ij}^{(k)}]_{n_k \times n_k}$, $2 \leq k \leq m$, in the block representation (3) of the matrix $A^+$ are computed as follows:

1) if $A_k$ is a block of type 1, then

$$h_{ij}^{(k)} = \frac{(-1)^{i+j+1}}{\Delta_{k-1}} \prod_{s=i}^{i-1} r_s^{(k)}$$

$$h_{ij}^{(k)} = 0 \quad \text{in the remaining cases};$$

2) if $A_k$ is a block of type 2, then

$$h_{i1}^{(k)} = \frac{1}{\Delta_{k-1}},$$

$$h_{ij}^{(k)} = 0 \quad \text{in the remaining cases};$$

3) if $A_m$ is a block of type 3 and $n_m = 1$, then

$$h_{1n_m}^{(m)} = \frac{\Delta_{m-1}}{d_1^{(m)} + \Delta_{m-1}}; \quad h_{1j} = 0, j = 1, 2, \ldots, n_m - 1;$$

4) if $A_m$ is a block of type 3 and $n_m \geq 2$, then

4a) for the indeces $j = 1, 2, \ldots, n_{m-1} - 1$ and $i = 1, 2, \ldots, n_m$:

$$h_{ij}^{(m)} = 0;$$

4b) for the indeces $j = n_m$ and $i = 1, 2, \ldots, n_m$:

$$h_{ij}^{(m)} = \frac{(-1)^{i+1} \Delta_{m-1}}{d_1^{(m)} + \Delta_{m-1}} \prod_{s=i}^{n_m} \frac{1}{r_s^{(m)}} + \frac{1}{d_k^{(m)} + \Delta_{m-1}} \sum_{k=1}^{n_m-1} \frac{1}{b_k^{(m)}} \left( \prod_{s=1}^{k-1} r_s^{(m)} \right) \left( \prod_{s=k+1}^{n_m} \frac{1}{r_s^{(m)}} \right) + D^{(m)},$$

where $D^{(m)} = \Delta_{m-1}^2 \prod_{s=1}^{n_m-1} r_s^{(m)}$.

Thus, in Theorems 1 and 2 we have derived closed form expressions for the entries of the Moore–Penrose inverse of upper bidiagonal matrix $A$ from (1). In the next section we discuss an issue of practical computation of the matrix $A^+$. 
A Procedure to Compute the Moore–Penrose Inverse. In the paper [2] we have developed a computational procedure of finding the first diagonal block $Z_1$ in the block representation (3) of the matrix $A^+$ (see Block $Z_1$, Procedure $Z1$). Further, in the paper [3] we have developed numerical algorithms to compute model matrices $Z$ and $H$ (see (10),(11) in [3]). Taking advantage of these results, below we give the following computational procedure.

**Procedure 2d/pinv** $(A,n \Rightarrow A^+)$

**Input:** an upper bidiagonal matrix $A$ of the form (1).

1. A partition (2) of the matrix $A$ into blocks according to the rule specified in [2](Introduction); identification of the blocks $A_k$ $(1 \leq k \leq m)$, $B_k (1 \leq k \leq m-1)$ and determination of the parameters $n_k$, $1 \leq k \leq m$, which define the block sizes. In each block $A_k$ its own internal numbering of the entries is given (see (8),(9)). The quantities $\Delta_k$, $1 \leq k \leq m-1$, are introduced (see (11)).

2. The block $Z_1$ in the block representation (3) of the matrix $A^+$ is computed. For that the procedure $Z1(A_1,n_1 \Rightarrow Z_1)$ from [2] is used. The procedure requires $n^2_1 + O(n_1)$ arithmetical operations.

   If $m = 1$, then the computations are completed and $A^+ = Z_1$.

   If $m \geq 2$, then proceed to successive computation of the blocks $Z_k$ and $H_k$, for $k = 2,3,\ldots,m$.

3. If $A_k$ is a block of type 1, the blocks $Z_k$ and $H_k$ are computed using the algorithm $Z,H/caseB (A_k,\Delta_{k-1},n_k,n_{k-1} \Rightarrow Z_k,H_k)$ given in [3]. The algorithm requires $(1/2)n^2_k + O(n_k)$ arithmetical operations.

4. If $A_k$ is a block of type 2, simple expressions for the entries of the blocks $Z_k$ and $H_k$ are obtained in Lemmas 5 and 6 of [3]:

   if $n_k = 1$, then
   
   $$Z_k = [0]_{1 \times 1}, \quad H_k = \left[ 0 \ldots 0 \frac{1}{\Delta_{k-1}} \right]_{1 \times n_k}$$

   if $n_k \geq 2$, then
   
   $$Z_k = \begin{bmatrix}
   0 & \cdots & 0 & b_{1}^{(k)-1} \\
   b_{1}^{(k)} & 0 & \cdots & 0 \\
   b_{2}^{(k)} & 0 & \cdots & 0 \\
   \vdots & \ddots & \ddots & \ddots \\
   b_{n_k-1}^{(k)} & 0 & \cdots & 0
   \end{bmatrix}, \quad H_k = \begin{bmatrix}
   0 & \cdots & 0 & 1 \\
   0 & \cdots & 0 & \Delta_{k-1} \\
   \vdots & \ddots & \ddots & \ddots \\
   0 & \cdots & 0 & 0
   \end{bmatrix}.$$  

   It requires no more than $n_k$ arithmetical operations.

5. If $A_m$ is a block of type 3, the blocks $Z_m$ and $H_m$ are computed using the algorithm $Z,H/caseA (A_m,\Delta_{m-1},n_m,n_{m-1} \Rightarrow Z_m,H_m)$ given in [3]. The algorithm requires $n^2_m + O(n_m)$ arithmetical operations.

**Output:** matrix $A^+$.

End procedure.
Direct calculations show that for $m \geq 2$ the described computational procedure requires no more than

$$n_1^2 + \frac{1}{2} \sum_{k=2}^{m-1} n_k^2 + n_m^2 + O(n)$$

arithmetical operations (recall that $n_1 + n_2 + \cdots + n_m = n$). If $m = 1$, this number does not exceed $n_1^2 + O(n_1)$.

Thus we can formulate the following statement.

**Proposition.** Let $A$ be a singular upper bidiagonal matrix of the form (1) with non-zero over-diagonal entries $b_1, b_2, \ldots, b_{n-1}$. Then the Moore–Penrose inverse $A^+$ of this matrix can be obtained using the computational procedure $2d/pinv$, which requires no more than $n_1^2 + O(n_1)$ (if $m = 1$) or

$$n_1^2 + \frac{1}{2} \sum_{k=2}^{m-1} n_k^2 + n_m^2 + O(n)$$

(if $m \geq 2$) arithmetical operations.

As a clarification, we note the following important features of the procedure. Proceeding from the structure of the blocks in the block representation (3) of the matrix $A^+$ (namely, the presence of zeros located at predetermined places) and estimation of the number of arithmetical operations required to compute each block, we can assert that for computing one non-zero entry of the matrix $A^+$ asymptotically is expended one arithmetical operation. Therefore the proposed method can be considered as an optimal.

**Concluding Remarks.** As a result of the study carried out we have obtained a solution to the problem of the Moore–Penrose inversion of singular upper bidiagonal matrices. We have derived a closed form expressions for the entries of pseudoinverse matrix and developed an optimal numerical algorithm for their computation.

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**REFERENCES**