PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2016, № 2, p. 63–66

COMMUNICATIONS

Mathematics

ON LEBESGUE CONSTANTS OF VILENKIN SYSTEMS

S. A. SARGSYAN *

Chair of Higher Mathematics, Department of Physics YSU, Armenia

In the paper some properties of Lebesgue constants $\left\{L_n(W)\right\}_{n=1}^{\infty}$ of Vilenkin system are investigated. Non almost convergence property for the sequence $\left\{\dfrac{L_n(W)}{\log_2 n}\right\}_{n=2}^{\infty}$ is obtained.

MSC2010: Primary 42C10, Secondary 42B08.

***Keywords*:** Vilenkin system, Lebesgue constants, almost convergence.

Introduction. Let $\Psi = \{\psi_k(x)\}_{k=1}^{\infty}$ be an orthonormal system of functions defined on $[a,b]$. The Lebesgue constants of Ψ is defined as follows:

$$L_n(\Psi,x) := \int_a^b \left|\sum_{k=1}^{n} \psi_k(x)\overline{\psi_k(t)}\right| dt \quad \text{(here } \bar{x} \text{ means the complex conjugate of } x\text{)}.$$

If this functions are independent on x, then they are called Lebesgue constants $\{L_n(\Psi)\}_{n=1}^{\infty}$ of system Ψ. Recall the definition of Vilenkin systems. Consider an arbitrary sequence of natural numbers $P \equiv \{p_1, p_2, \ldots, p_k, \ldots\}$, where $p_j \geq 2$ for all $j \in \mathbb{N}$.

Let denote $m_0 = 1$, $m_k = \prod_{j=1}^{k} p_j \quad (p_j \geq 2)$.

It is easy to see that for each $x \in [0,1)$ and for each $n \in \mathbb{N}$ there exist integers $x_j, \alpha_j \in \left\{0,1,\ldots\ p_j-1\right\}$ $\left(\text{in the case } x = \dfrac{l}{m_k},\ l \in \mathbb{N},\ 0 \leq l \leq m_k - 1, \text{ we take } x_j = 0\right.$ $\left.\text{for all } j > k\right)$, so that $n = \sum_{j=1}^{\infty} \alpha_j m_{j-1}$ and $x = \sum_{j=1}^{\infty} \dfrac{x_j}{m_j}$ (P-adic expansions).

* E-mail: stepansargsyan@ysu.am

Vilenkin or multiplicative system with respect to the sequence P is defined as follows:

$$W_0(x) \equiv 1;\ W_n(x) = \exp\left(2\pi i \sum_{j=1}^{k} \alpha_j \frac{x_j}{p_j}\right).$$

Obviously the nth function can be represented by

$$W_n(x) = \prod_{j=1}^{k} (W_{m_{j-1}}(x))^{\alpha_j}.$$

Note that systems corresponding to distinct sequences $P = \{p_k\}$ are different, in particular, if $P \equiv \{2, 2, \ldots, 2, \ldots\}$ the Vilenkin system coincides with the Walsh one (see [1]). The theory of these systems was developed by N. Ya. Vilenkin in 1946 (see [2, 3]).

The Lebesgue constants of Vilenkin system have the form

$$L_n(W) = \int_0^1 |D_n(t)|dt, \tag{1}$$

where $D_n(t) = \sum_{k=0}^{n-1} W_k(t)$ is the nth Dirichlet kernel of Vilenkin system.

It is known [4] that $\lim\limits_{n\to\infty} \dfrac{L_n(T)}{\ln n} = \dfrac{4}{\pi}$, where T is the trygonometric system. Note that, in contrast to this, for the Vilenkin system it was proved [3] that

$$0 = \liminf_{n\to\infty} \frac{L_n(W)}{\log_2 n} < \limsup_{n\to\infty} \frac{L_n(W)}{\log_2 n} < \infty. \tag{2}$$

Recall that a bounded sequence $\{x_n\}_{n=1}^{\infty}$ is called almost convergent, if for some $a \in \mathbb{R}$ we have $\lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=m+1}^{m+n} x_k = a$ uniformly by m. Denote

$$q(x_n) = \lim_{n\to\infty} \inf_{m\in\mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k \text{ and } p(x_n) = \lim_{n\to\infty} \sup_{m\in\mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k.$$

This limits exist for every bounded sequence, and obviously the almost convergence of a sequence $\{x_n\}_{n=1}^{\infty}$ is equivalent to the condition $q(x_n) = p(x_n)$ (see [5, 6]).

In this paper we prove the following

Theorem 1. For any Vilenkin system the following equivalencies are true:

1. $q\left(\dfrac{L_n(W)}{\log_2 n}\right) = 0 \ \left(= \liminf\limits_{n\to\infty} \dfrac{L_n(W)}{\log_2 n}\right);$
2. $p\left(\dfrac{L_n(W)}{\log_2 n}\right) = \limsup\limits_{n\to\infty} \dfrac{L_n(W)}{\log_2 n} < \infty.$

From this Theorem as a direct consequence we obtain that the sequence $\left\{\dfrac{L_n(W)}{\log_2 n}\right\}_{n=2}^{\infty}$ is not almost convergent. Note that the analogues result for the Walsh system is formulated in [6].

Auxiliary Propositions. We will use the following properties of Vilenkin system

$$L_{m_k}(W) = 1,\ k = 0, 1, \ldots, \tag{3}$$

$$W_{lm_k+\beta}(x) = W_{lm_k}(x) W_\beta(x),\ \text{ if } \ \beta < m_k\ (k, l, \beta \in \mathbb{N}). \tag{4}$$

From (4) we get

$$\begin{gathered} D_{m_k+r}(t) \equiv \sum_{j=0}^{m_k-1} W_j(t) + W_{m_k}(t) \sum_{j=0}^{r-1} W_j(t) \equiv \\ \equiv D_{m_k}(t) + W_{m_k}(t) D_r(t) \ \text{ for all } \ 1 < r \leq m_k. \end{gathered} \tag{5}$$

Proof of Main Result. Let us begin with a proof of first equation.

We put

$$l_n := \frac{L_n(W)}{\log_2 n},\ \check{l_n} = \inf_{m\in\mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} l_k \ \text{ and } \ \hat{l_n} = \sup_{m\in\mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} l_k.$$

Let $\varepsilon > 0$ be an arbitrary positive number and $n \in \mathbb{N}$. Obviously there exists $k \in \mathbb{N}$ depending on n such that

$$m_k > n \ \text{ and } \ \frac{1}{k}\left(\frac{1}{n}\sum_{r=1}^{n}(1 + L_r)\right) < \varepsilon. \tag{6}$$

From (1), (3) and (5) we obtain

$$L_{m_k+r} \leq 1 + L_r \quad \text{for all} \quad 1 \leq r \leq n.$$

Combining this with (6) and taking into consideration a relation $m_k \geq 2^k$, we get

$$\frac{1}{n}\sum_{j=m_k+1}^{m_k+n} l_j \leq \frac{1}{k}\left(\frac{1}{n}\sum_{r=1}^{n}(1 + L_r)\right) < \varepsilon.$$

Therefore, we have $0 \leq \check{l_n} < \varepsilon$ for any natural number n, and eventually since ε is arbitrary we get the first equation $q(l_n) = 0$.

Next we prove that $\lim\limits_{r\to\infty} \hat{l_r} \geq c$. We put

$$c = \limsup_{n\to\infty} l_n,\ \ c_1 = \sup_{n\in\mathbb{N}} l_n. \tag{7}$$

Let r be any natural number and $\varepsilon > 0$. We fix $k_0 \in \mathbb{N}$ such that $m_{k_0} \geq r$, then we take $n_0 \in \mathbb{N}$ $(n_0 > m_{k_0})$, so that

$$l_{n_0} > c - \frac{\varepsilon}{3},\ \frac{\log_2 m_{k_0}}{\log_2(n_0 - m_{k_0})} < \frac{\varepsilon}{6c_1} \ \text{ and } \ \frac{\log_2 n_0}{\log_2(n_0 + m_{k_0})} > 1 - \frac{\varepsilon}{3(c+1)}. \tag{8}$$

The P-adic expansion of n_0 has the form $n_0 = \sum\limits_{j=0}^{k} \alpha_j m_j$. Denote

$$n_0' := \sum_{j=k_0}^{k} \alpha_j m_j. \tag{9}$$

Let $n \in [n_0', n_0' + m_{k_0})$. By the same argument as in (5) we get

$$D_n(t) \equiv D_{n_0'}(t) + W_{n_0'}(t) D_{n-n_0'}(t).$$

Thus, from (1) and reverse triangle inequality, we obtain

$$L_n \geq L_{n'_0} - L_{n-n'_0} \quad \text{and} \quad L_{n'_0} \geq L_{n_0} - L_{n-n'_0}.$$

Hence (see also (7)),

$$L_n \geq L_{n_0} - 2L_{n-n'_0} \geq L_{n_0} - 2c_1 \log_2 m_{k_0}.$$

From this and (8), (9) we have

$$l_n \geq \frac{L_{n_0}}{\log_2 n_0}\left(\frac{\log_2 n_0}{\log_2 n}\right) - 2\frac{c_1 \log_2 m_{k_0}}{\log_2 n} > c - \varepsilon. \tag{10}$$

From (10) we get

$$\frac{1}{r}\sum_{n=n'_0+1}^{n'_0+r} l_n > c - \varepsilon,$$

which implies $\hat{l}_r \geq c - \varepsilon$, and from arbitrariness of r and ε we obtain that $\lim\limits_{r\to\infty} \hat{l}_r \geq c$. It remains to show that $\lim\limits_{r\to\infty} \hat{l}_r \leq c$.

Again we take $\varepsilon > 0$ to be an arbitrary number, then we choose natural numbers k_0 and $k'_0 > k_0$ such that

$$l_k < c + \frac{\varepsilon}{2} \quad \text{for all} \quad k \geq k_0 \quad \text{and} \quad \frac{1}{k'_0}\sum_{k=1}^{k_0} l_k < \frac{\varepsilon}{2}. \tag{11}$$

If $m < k_0$ or all $r \geq k'_0$, we have

$$\frac{1}{r}\sum_{k=m+1}^{m+r} l_k = \frac{1}{r}\sum_{k=m+1}^{k_0} l_k + \frac{1}{r}\sum_{k=k_0+1}^{m+r} l_k\ . \tag{12}$$

From (11) and (12) we get

$$\frac{1}{r}\sum_{k=m+1}^{m+r} l_k \leq c + \varepsilon \quad \text{for all} \quad r \geq k'_0.$$

If $m \geq k_0$, then from (11) for all $r \in \mathbb{N}$ we obtain

$$\frac{1}{r}\sum_{k=m+1}^{m+r} l_k \leq c + \frac{\varepsilon}{2}.$$

Hence, $\hat{l}_r \leq c + \varepsilon$ for all $r \geq k'_0$ and eventually we get $\lim\limits_{r\to\infty} \hat{l}_r \leq c$. □

Received 08.02.2016

REFERENCES

1. **Golubov B.I., Efimov A.V., Skvortsov V.A.** Walsh Series and Transforms: Theory and Applications. Dordrecht: Kluwer Academic Publishers, 1991.
2. **Agaev G.N., Vilenkin N.Ya., Dzhafarli G.M., Rubinshtein A.I.** Multiplicative Systems of Functions and Harmonic Analysis on Zero-Dimensional Groups. Baku: ELM, 1981 (in Russian).
3. **Vilenkin N.Ya.** On a Class of Complete Orthonormal Systems. // Izv. Akad. Nauk SSSR, 1947, v. 11, p. 363–400 (in Russian).
4. **Bari N.K.** Trigonometric Series. M.: Fizmatlit, 1961 (in Russian).
5. **Sucheston L.** Banach Limits. // Amer. Math. Monthly, 1967, v. 74, p. 308–311.
6. **Astashkin S.V., Semenov E.M.** Lebesgue Constants of the Walsh System. // Doklady Akademii Nauk, ser. Matematica, 2015, v.91, № 3, p. 344–346.