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P h y s i c s

COULOMB SYSTEMS WITH CALOGERO INTERACTION

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We describe the integrals of motion of the high-dimensional Coulomb system with and without the Stark term, perturbed by the Calogero interaction.

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Introduction. Consider the rational Calogero model, which describes one-dimensional particles, interacting with the inverse-square potential [1]

$$\mathcal{H}_0 = \sum_i \frac{p_i^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2}. \tag{1}$$

Apart from the N Liouville integrals, it possesses also $N-1$ additional constants of motion [2]. As a result, the system is maximally superintegrable. It possesses various integrable generalizations with physical applications [3, 4]. Note that the high-dimensional oscillator and Coulomb systems are the most known nontrivial superintegrable systems with second order integrals in momentum. The Calogero system is more complicate, since most of its integrals have higher order in momemtum.

The mixture of the Coulomb and Calogero potentials gives rise to a more general integrable N-dimensional system [5]. Recently we have shown that the Calogero–Coulomb system is also superintegrable [6]. An explicit form of the complete set of constants of motion can be derived by taking proper deformations of the corresponding integrals of the underlying Coulomb system, then forming the symmetric polynomials on them [6, 7]. This method differs from the standard construction, so that the deformations of the Liouville integrals do not commute any more. Nevertheless, the functional independence of the constricted integrals of motion is preserved.

In this paper we briefly describe the N-dimensional Coulomb problem with the additional Calogero potential [7], as well as its extension in the uniform constant electric field given by the Stark potential [8]. We refer them shortly as the Calogero–Coulomb and Calogero–Coulomb–Stark problems, correspondingly. Below we demonstrate that these systems possess the deformed hidden symmetries, inherited from the well-known Rungle–Lenz vector in the Coulomb problem.

Calogero–Coulomb Problem. The Calogero–Coulomb problem is a mixture of the N-particle rational Calogero model (1) and of the N-dimensional Coulomb system [5]:

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$$\mathcal{H}_\gamma = \frac{\boldsymbol{p}^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2} - \frac{\gamma}{r}. \tag{2}$$

It inherits most of the properties of the original Coulomb system and possesses hidden symmetries given by an analog of Runge–Lenz vector [6, 7]. It is convenient to describe this system by means of the Dunkl operators which make transparent the analogy with the initial Coulomb problem.

Let us consider instead the extended Hamiltonian

$$\mathcal{H}_\gamma^{\text{gen}} = \frac{\boldsymbol{\pi}^2}{2} - \frac{\gamma}{r} = \frac{\boldsymbol{p}^2}{2} + \sum_{i<j} \frac{g(g - s_{ij})}{(x_i - x_j)^2} - \frac{\gamma}{r}. \tag{3}$$

The modified momentum is expressed in terms of the Dunkl operators by

$$\boldsymbol{\pi} = -\imath \boldsymbol{\nabla}, \qquad \nabla_i = \partial_i - \sum_{j \neq i} \frac{g}{x_i - x_j} s_{ij}. \tag{4}$$

The operator s_{ij} permutes the i-th and j-th coordinates. On the symmetric wavefunctions the generalized Hamiltonian $\mathcal{H}_\gamma^{\text{gen}}$ reduces to the Calogero–Coulomb Hamiltonian (2).

The Dunkl operators commute mutually like ordinary partial derivatives. However, their commutations with coordinates are nontrivial deformations of the Heisenberg algebra relations [9]

$$[\pi_i, x_j] = -\imath S_{ij}. \tag{5}$$

The operators S_{ij} for $i \neq j$ are just rescaled permutations:

$$S_{ij} = -g s_{ij}. \tag{6}$$

In additions, the S_{ii} are defined by the relation

$$\sum_j S_{ij} = 1. \tag{7}$$

Constants of Motion of $\mathcal{H}_\gamma$. Define now the deformed angular momentum operator via the Dunkl momentum [10, 11]:

$$L_{ij} = x_i \pi_j - x_j \pi_i. \tag{8}$$

It preserves the generalized Calogero–Coulomb Hamiltonian and satisfies the deformed angular momentum commutation relations [10].

The deformed Runge–Lenz vector, preserving as well the same Hamiltonian, reads [7]

$$A_i = \frac{1}{2} \sum_j \{L_{ij}, \pi_j\} + \frac{\imath}{2}[\pi_i, S] - \frac{\gamma x_i}{r}. \tag{9}$$

It contains the permutation-group invariant element, which vanishes in the absence of the Calogero term

$$S = \sum_{i<j} S_{ij}. \tag{10}$$

The Calogero–Coulomb problem can be obtained by the restriction of the extended Hamiltonian (3) to the symmetric wavefunctions. Therefore, its constants of motion can be constructed by taking the symmetric polynomials on the components of the Dunkl angular momentum and Runge–Lenz vector [6, 7]:

$$\mathcal{L}_{2k} = \sum_{i<j} L_{ij}^{2k}, \qquad \mathcal{A}_k = \sum_i A_i^k. \tag{11}$$

The expressions above demonstrate that the Calogero–Coulomb problem is a superintegrable system, like the pure Calogero [2] and Coulomb models. Note that the the square of Dunkl angular momentum is related to the the angular part $\mathcal{I}$ of the Calogero Hamiltonian [10]:

$$\mathcal{L}_2 = 2\mathcal{I} + S(S - N + 2). \tag{12}$$

In two dimension the symmetries of the Calogero–Coulomb system, based on the dihedral group D_2, have been studied also in [12].

Coulomb–Calogero–Stark Problem. Consider the N-dimensional Coulomb problem in constant electric field F in the presence of the Calogero interaction:

$$\mathcal{H}_{\gamma,F} = \frac{\boldsymbol{p}^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2} - \frac{\gamma}{r} + F x_0, \tag{13}$$

where x_0 is the normalized center-of-mass coordinate (see Eq. (16) below). The external field is aligned in the direction $(1,1,\dots,1)$, which ensures the permutation invariance of the Hamiltonian. In the absence of the external field, this model is reduced to the Calogero–Coulomb model, considered above.

The generalized Hamiltonian is defined in terms of the Dunkl momentum (4) as follows:

$$\mathcal{H}^{\text{gen}}_{\gamma,F} = \frac{\pi^2}{2} - \frac{\gamma}{r} + F x_0. \tag{14}$$

The entire Dunk angular momentum tensor (8) is not an integral of motion any more. Instead, its components, which are orthogonal to the external field, are preserved,

$$L^{\perp}_{ij} = L_{ij} + \frac{1}{N} \sum_{k} (L_{jk} - L_{ik}). \tag{15}$$

Alternatively, one can express them in terms of the Jacobi coordinates, which separate the center-of-mass from the relative motion. They are defined by the orthogonal map [13, 14]

$$x_0 = \frac{1}{\sqrt{N}}(x_1 + \dots + x_N), \qquad \tilde{x}_k = \frac{1}{\sqrt{k(k+1)}}(x_1 + \dots + x_k - k x_{k+1}), \tag{16}$$

where $1 \le k \le N-1$. The first coordinate describes the center of mass, while the others, marked by tilde, characterize the relative motion.

Constants of Motion of $\mathcal{H}_{\gamma,F}$. Denote now by $\tilde{L}_{ij}$ the components of the deformed relative angular momentum, rotated by the Jacobi transformation. The algebra generated by $L^{\perp}_{ij}$, in fact, coincides with the $\tilde{L}_{ij}$, which are responsible for the relative motion ($1 \le i$, $j \le N-1$). In the absence of Calogero interaction they form the $SO(N-1)$ subalgebra, which describes the rotations in the hyperspace, orthogonal to the center-of-mass direction.

Apart from the deformed relative angular momenta, the modified component of the Runge–Lenz vector (9) along the field direction is preserved as well. It reads

$$A = x_0 \Big(2\mathcal{H}^{\text{gen}}_{\gamma,F} + \frac{\gamma}{r} \Big) - \Big(r p_r + \frac{N-1}{2\imath} \Big) p_0 - \frac{F}{2} \left(r^2 + 3x_0^2 \right). \tag{17}$$

This invariant commutes with the deformed relative angular momentum.

In the $g = 0$ limit, one can extract from these symmetry generators the standard Liouville integrals of the Coulomb–Stark system. The first $N-2$ integrals can be chosen to be the quadratic Casimir elements of the naturally embedded algebras

$$SO(2) \subset SO(3) \subset \ldots \subset SO(N-1). \tag{18}$$

They are described in the relative angular coordinates and momenta. The last two integrals are given by the Hamiltonian and the modified component of the Runge–Lenz vector, which had been constructed for $N = 3$ in [15].

Out of the $g = 0$ point, we deal with the deformed quantities, and the Liouville property can not be extended straightforwardly. Nevertheless, in the presence of a constant uniform electric field, the generalized Calogero–Coulomb model (14) still remains integrable.

The integrals of the pure Calogero–Coulomb system (13) obtained by the restriction to the symmetric wavefunctions, must be symmetric, too. Since the longitudinal component of the Runge–Lenz vector (17) obeys this condition, it remains as a correct integral for this system:

$$[A,\mathcal{H}_{\gamma,F}]=0. \tag{19}$$

We should take symmetric expressions of the kinematical constants of motion, too, as in the absence of the electric field [7]. For this purpose it is more suitable to use the angular momentum in Jacobi coordinates:

$$[\mathcal{H}_{\gamma,F},\tilde{\mathcal{L}}_{2k}]=0, \qquad \tilde{\mathcal{L}}_{2k}=\sum_{1\le i<j\le N-1}\tilde{L}_{ij}^{2k}. \tag{20}$$

The first member of this family is the square of the relative Dunkl angular momentum. It is related to the angular part of the Calogero model with reduced center of mass $\tilde{\mathcal{I}}$, which we call the relative angular Calogero Hamiltonian, by the same formula as Eq. (12) above. So, we have proved the integrability of the Calogero–Coulomb–Stark system.

Separation of Variables in Parabolic Coordinates in $\mathcal{H}_{\gamma,F}$. It is well known that the Coulomb–Stark system admits separation of variables in parabolic coordinates. It appears that the Calogero–Coulomb–Stark system admits complete separation of variables in parabolic coordinates for $N=2,3$ and partial separation for $N>3$ [8].

In the Jacobi coordinates (16), the last system acquires the following form:

$$\mathcal{H}_{\gamma,F}=\frac{p_0^2}{2}-\frac{\gamma}{\sqrt{x_0^2+\tilde{x}^2}}+Fx_0+\tilde{\mathcal{H}}_0, \tag{21}$$

where the last term is the Calogero Hamiltonian (1) with reduced center of mass. We pass to the parabolic coordinates (ξ,η,φ_ι), where φ_ι are the relative angular variables, and

$$\xi=r+x_0, \qquad \eta=r-x_0. \tag{22}$$

In new coordinates the Hamiltonian (21) is expressed as follows:

$$\mathcal{H}_{\gamma,F}=-\frac{2}{\xi+\eta}\left(\gamma+B_\xi+B_\eta\right)+\frac{\tilde{\mathcal{I}}}{\xi\eta}+\frac{F}{2}(\xi-\eta), \tag{23}$$

where we have shorten the kinetic term using the notation

$$B_\xi=\frac{1}{\xi^{\frac{N-3}{2}}}\frac{\partial}{\partial\xi}\xi^{\frac{N-1}{2}}\frac{\partial}{\partial\xi}. \tag{24}$$

Further we proceed by extending straightforwardly the steps, applied for the usual Coulomb system in external field in [16]. Employing the following ansatz to the total wavefunction

$$\Psi(\xi,\eta,\varphi_\iota)=\Phi_1(\xi)\Phi_2(\eta)\psi(\varphi_\iota), \tag{25}$$

we decouple Schrödinger equation

$$\mathcal{H}_{\gamma,F}\Psi=E\Psi \tag{26}$$

into three parts. The two of them depend, respectively, on ξ and η,

$$\left(B_\xi+\frac{E}{2}\xi-\frac{F}{4}\xi^2-\frac{\tilde{q}(\tilde{q}+N-3)}{4\xi}+\lambda_1\right)\Phi_1(\xi)=0, \tag{27a}$$

$$\left(B_\eta+\frac{E}{2}\eta+\frac{F}{4}\eta^2-\frac{\tilde{q}(\tilde{q}+N-3)}{4\eta}+\lambda_2\right)\Phi_2(\eta)=0, \tag{27b}$$

where $\lambda_1+\lambda_2=\gamma$. The last equation describes the spectrum and eigenstates of the relative angular Calogero model [17]:

$$\tilde{\mathcal{I}}(\varphi_\iota,\partial_{\varphi_\iota})\psi_{\tilde{q}}(\varphi_\iota)=\frac{\tilde{q}(\tilde{q}+N-3)}{2}\psi_{\tilde{q}}(\varphi_\iota). \tag{27c}$$

In particular, the spectrum is determined by the numbers

$$\tilde{q} = \frac{gN(N-1)}{2} + 3l_3 + \ldots + Nl_N \qquad \text{with} \qquad l_i = 0, 1, 2, \ldots \tag{28}$$

For integer values of the coupling g, the angular energy spectrum is that of a free particle with angular momentum $\tilde{q}$ on the $(N-2)$-dimensional sphere, but has a significantly lower degeneracy due to the restriction to the symmetric wavefunctions [17].

The longitudinal component of the Runge–Lenz vector (17) separates the equations (27a) and (27b):

$$A\Psi = (\lambda_2 - \lambda_1)\Psi. \tag{29}$$

The second invariant, given by the relative angular Hamiltonian $\tilde{\mathfrak{I}}$, is common in both cases and separates the relative angular degrees of freedom. As in the usual Coulomb problem [18], the electric field completely removes the degeneracy in the orbital momentum, but preserves the degeneracy with respect to q.

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REFERENCES

1. **Calogero F.** Solution of a Three-Body Problem in One Dimension. // J. Math. Phys., 1969, v. 10, p. 2191. Solution of the One-Dimensional N-Body Problems with Quadratic and/or Inversely Quadratic Pair Potentials. // ibid., 1971, v. 12, p. 419.
2. **Wojciechowski S.** Superintegrability of the Calogero–Moser System. // Phys. Lett. A, 1983, v. 95, p. 279.
3. **Olshanetsky M., Perelomov A.** Classical Integrable Finite Dimensional Systems Related to Lie Algebras. // Phys. Rept. 1981, v. 71, p. 313.
4. **Polychronakos A.P.** Physics and Mathematics of Calogero Particles. // J. Phys. A., 2006, v. 39, p. 12793.
5. **Khare A.** Exact Solution of an N-Body Problem in One Dimension. // J. Phys. A, 1996, v. 29, p. L45.
6. **Hakobyan T., Lechtenfeld O., Nersessian A.** Superintegrability of Generalized Calogero Models with Oscillator or Coulomb Potential. // Phys. Rev. D., 2014, v. 90, p. 101701(R).
7. **Hakobyan T., Nersessian A.** Runge–Lenz Vector in Calogero–Coulomb Problem. // Phys. Rev. A, 2015, v. 92, p. 022111.
8. **Hakobyan T., Nersessian A.** Integrability and Separation of Variables in Calogero–Coulomb–Stark and Two-Center Calogero–Coulomb Systems. // Phys. Rev. D, 2016, v. 93, p. 045025.
9. **Dunkl C.F.** Differential-Difference Operators Associated to Reflection Groups. // Trans. Amer. Math. Soc., 1989, v. 311, p. 167.
10. **Feigin M., Hakobyan T.** On Dunkl Angular Momenta Algebra. // JHEP, 2015, v. 11, p. 107.
11. **Feigin M.** Intertwining Relations for the Spherical Parts of Generalized Calogero Operators. // Theor. Math. Phys., 2003, v. 135, p. 497.
12. **Genest V.X., Lapointe A., Vinet L.** The Dunkl–Coulomb Problem in the Plane. // Phys. Lett. A., 2015, v. 379, p. 923.
13. **Hakobyan T., Nersessian A., Yeghikyan V.** Cuboctahedric Higgs Oscillator from the Calogero Model. // J. Phys. A., 2009, v. 42, p. 205206.
14. **Hakobyan T., Lechtenfeld O., Nersessian A.** The Spherical Sector of the Calogero Model as a Reduced Matrix Model. // Nucl. Phys. B., 2012, v. 858, p. 250.
15. **Redmond P.J.** Generalization of the Runge–Lenz Vector in the Presence of an Electric Field. // Phys. Rev. B., 1964, v. 133, p. 1352.
16. **Helfrich K.** Constants of Motion for Separable One-Particle Problems with Cylinder Symmetry. // Theoret. Chim. Acta (Berl.), 1972, v. 24, p. 271.
17. **Feigin M., Lechtenfeld O., Polychronakos A.** The Quantum Angular Calogero–Moser Model. // JHEP, 2013, v. 1307, p. 162.
18. **Landau L., Lifshitz E.** Quantum Mechanics. Oxford: Pergamon Press, 1977.