PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2016, № 3, p. 25–31

Physics

ODD SYMMETRIC TENSORS, AND AN ANALOGUE OF THE LEVI-CIVITA CONNECTION FOR ODD SYMPLECTIC STRUCTURE

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We consider odd Poisson (odd symplectic) structure on supermanifolds induced by an odd symmetric rank 2 (non-degenerate) contravariant tensor field. We describe the difference between odd Riemannian and odd symplectic structure in terms of the Cartan prolongation of the corresponding Lie algebras, and formulate an analogue of the Levi-Civita theorem for an odd symplectic supermanifold.

Keywords: odd Poisson bracket, half-density, odd (anti)symmetric tensor, Cartan prolongation, second order compensation field, odd symplectic geometry, odd canonical operator.

Introduction. The role of an odd symmetric rank 2 contravariant tensor field on a supermanifold is twofold. On one hand, it may be considered as the principal symbol of an odd second order differential operator; on the other, the tensor field defines an odd bracket on functions, an odd Poisson bracket, if the bracket satisfies a Jacobi identity. It is very illuminating to utilise this versatility to study the properties of odd brackets in terms of odd second order operators (see [1, 2]).

The fact that the study of odd brackets and odd operators may be combined does not have an analogy in usual mathematics (where by "usual" we mean without anticommuting variables). In the usual setting, symmetric tensors are related with Riemannian geometry and second order operators, whereas antisymmetric tensors are related with Poisson and symplectic geometry. In supermathematics the notion of symmetry for tensors becomes more subtle and the difference between odd Riemannian and odd symplectic structures has to be based on the fact that a Riemannian structure is "rigid", whilst a symplectic structure is "soft", i.e. the space of infinitesimal isometries (the Killing vector fields) of a Riemannian structure is finite dimensional, and the space of infinitesimal isometries (the Hamiltonian vector fields) of a symplectic structure is infinite dimensional. This fundamental difference is a consequence of the difference in the Cartan prolongation of the corresponding Lie algebras. When anticommuting variables are present, it is this characteristic feature that must be used to distinguish between these two structures. We discuss this phenomenon in the next section.

In the last section we study properties of a second order compensating field which naturally arises on odd Poisson supermanifolds. In the case when the Poisson structure is non-degenerate (an odd symplectic structure), this compensating field may be defined uniquely in a way analogous to the Levi-Civita connection.

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What Distinguishes Riemannian and Symplectic Structure? Recall the following textbook facts: Riemannian geometry and second order operators are yielded by symmetric rank 2 contravariant tensor fields, and Poisson and symplectic geometry arise from antisymmetric tensors. In more detail; a symmetric tensor field $G = g^{ik}(x)\partial_k \otimes \partial_i$ ($g^{ik}(x) = g^{ki}(x)$) can be considered as the principal symbol of a second order operator $\Delta = \frac{1}{2} g^{ik}(x)\partial_k\partial_i + \cdots$, and, if the tensor is non-degenerate and positive-definite, defines a Riemannian metric $g_{ik}(x)dx^k \otimes dx^i$. A rank 2 antisymmetric tensor field $P = P^{ik}(x)\partial_k \wedge \partial_i$ ($P^{ik} = -P^{ki}$) defines a bracket on functions on the manifold M: $\{f,g\} = -\{g,f\} = \partial_i f P^{ik}\partial_k g$, which is a Poisson bracket if P obeys the Jacobi identity: $P^{ir}\partial_r P^{jk} + \text{cyclic permutations} = 0$. If the Poisson structure is non-degenerate, the inverse tensor field $\omega_{ik}(x)dx^k \wedge dx^i$ ($\omega_{ik} = (P^{-1})_{ik}$) defines a symplectic structure on M.

What happens in the case when M is a supermanifold? In the same way one can consider a symmetric rank 2 contravariant tensor field,

$$\mathbf{E} = E^{AB}(z)\frac{\partial}{\partial z^B} \otimes \frac{\partial}{\partial z^A}, \quad E^{BA} = (-1)^{p(B)p(A)}E^{AB},$$

which may be considered as the principal symbol of a second order operator

$$\Delta = \frac{1}{2}E^{AB}(z)\frac{\partial}{\partial z^B} \cdot \frac{\partial}{\partial z^A} + \cdots$$

We choose local coordinates $z^A = (x^a, \theta^\alpha)$ on M, where x^a are even (bosonic) coordinates, their parity p is $p(x^a) = 0$, and θ^α are odd (fermionic) coordinates with parity $p(\theta^\alpha) = 1$. Odd coordinates anticommute: $x^a x^b = x^b x^a$, $\quad x^a\theta^\alpha = \theta^\alpha x^a$, $\quad$ whereas $\quad \theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha$.

Respectively for derivatives,

$$\frac{\partial}{\partial z^A} \cdot \frac{\partial}{\partial z^B} = (-1)^{p(A)p(B)}\frac{\partial}{\partial z^B} \cdot \frac{\partial}{\partial z^A}, \tag{1}$$

where we denote by $p(A)$ the parity of coordinate $z^A = (x^a, \theta^\alpha)$.

We have to distinguish two cases: when the tensor field $\mathbf{E} = E^{AB}\partial_B \otimes \partial_A$ is even and when it is odd. Consider for example a $p|q \times p|q$ matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A is a $p \times p$ matrix and B is a $q \times q$ matrix, both containing even entries. A tensor field defined by this matrix is an even tensor field on $p|q$-dimensional superspace $\mathbb{R}^{p|q}$. It is a symmetric field, if A is symmetric and B is antisymmetric, and vice versa, the tensor field is antisymmetric, if A is an antisymmetric matrix and B is a symmetric matrix.

Now let K and L be two $n \times n$ matrices with even entries. Then a tensor field defined by the $n|n \times n|n$ matrix $\begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$ is an example of an odd tensor field. This field is symmetric (antisymmetric), if $K = L$ ($K = -L$). An important case of this is the following: for the $n \times n$ unity matrix I, consider two $n|n \times n|n$ matrices:

$$S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2}$$

Matrix S defines an odd symmetric contravariant tensor field, whilst matrix G defines an odd antisymmetric contravariant tensor field. Later we will see that S leads to odd symplectic geometry and G to odd Riemannian geometry.

In the same way as for usual manifolds, an even symmetric tensor field yields Riemannian structure and an even antisymmetric tensor field obeying the Jacobi identity yields Poisson structure (see for details [1]). This is not the case for odd tensors where the notion of symmetry becomes more subtle.

Statement. *An odd contravariant symmetric tensor field (obeying the Jacobi identity) defines an odd Poisson structure. An odd contravariant antisymmetric tensor field obeying a non-degeneracy condition defines an odd Riemannian structure.*

Discuss this statement.

Example 1. Consider the $n|n$-dimensional superspace $\mathbb{R}^{n|n}$ with coordinates $(x^1,\dots,x^n|\theta_1,\dots,\theta_n)$, together with the odd symmetric rank 2 contravariant tensor field defined by the matrix S in equation (2). The components of S are constants and hence the Jacobi identity is fulfilled. We come to an odd non-degenerate Poisson bracket (symplectic structure) defined by the relations

$$\{x^i,\theta_j\}=\delta^i_j \quad \text{and} \quad \{x^i,x^k\}=0=\{\theta_i,\theta_k\}.$$

For arbitrary functions,

$$\{f,g\}=\frac{\partial f}{\partial x^i}\cdot\frac{\partial g}{\partial \theta_i}+(-1)^{p(f)}\frac{\partial f}{\partial \theta_i}\cdot\frac{\partial g}{\partial x^i}\,.$$

(The coordinates x^i,θ_j are Darboux coordinates for this symplectic structure.)

What is the reason for this paradoxical change in symmetry? In usual mathematics, given a non-degenerate (anti)symmetric tensor, its inverse tensor is also (anti)symmetric. This is no longer the case when passing to the super setting, where the symmetry of the inverse tensor is now dependent on the parity.

Example 2. Consider a rank 2 contravariant tensor field L^{AB} with L_{AB} its inverse: $L^{AB}L_{BC}=\delta^A_C$ (if it exists). Then one sees that

$$L^{AB}=\pm(-1)^{p(A)p(B)}L^{BA} \quad \Rightarrow \quad L_{AB}=\mp(-1)^{(p(A)+1)(p(B)+1)+p(L)}L_{BA}\,. \tag{3}$$

Consider also the map of tensors $X\mapsto\tilde{X}$,

$$X^{AB}\mapsto\tilde{X}^{AB}=(-1)^{p(A)}X^{AB}\,. \tag{4}$$

If $X^{AB}=\pm(-1)^{p(B)p(A)}X^{BA}$ then $\tilde{X}^{AB}=\mp(-1)^{(p(B)+1)(p(A)+1)}\tilde{X}^{BA}$. For example, if L is an odd non-degenerate symmetric (antisymmetric) tensor, then its inverse is also symmetric (antisymmetric), but *with respect to a shift in parity*, and the inverse to the tensor $\tilde{L}$ is an odd antisymmetric (symmetric) tensor with respect to usual parity.

This symmetry shift can be explained by the parity reversal functor $\Pi: V\to\Pi V$ (which reverses the parity in a vector space V). This functor defines a canonical isomorphism $V\otimes V\to\Pi V\otimes\Pi V$ (see Eq. (4)), which induces a canonical isomorphism $S^2(\Pi V)\cong\Pi^2\wedge{}^2V$ between the symmetric square of ΠV and the wedge square of V. When L is odd, L defines an isomorphism between T^*M and ΠTM, which induces a shift of symmetry (for details see the appendix of the article [3].)

Symmetry and antisymmetry cease to be the "wall" between odd symplectic and odd Riemannian geometry. In order to distinguish between the structures one has to consider other differences. Recall: Riemannian geometry possesses only a finite dimensional space of infinitesimal isometries (Killing vector fields), and symplectic geometry possesses an infinite dimensional space of infinitesimal isometries (Hamiltonian vector fields; each induced from a Hamiltonian function). Algebraically, this difference is expressed in terms of the difference in the Cartan prolongation of the orthogonal and symplectic Lie algebras.

Definition 1. Let $\mathcal{G}$ be a subalgebra of the linear Lie algebra $gl(n,\mathbb{R})$. The k^{th} Cartan prolongation of $\mathcal{G}$ is the space $\mathcal{G}_k$ $(k=0,1,2,\dots)$ of symmetric $k+1$-linear maps $l:\mathbb{R}^n\times\cdots\times\mathbb{R}^n\to\mathbb{R}^n$ such that for every k vectors $v_1,\dots,v_k$, the linear map

$$\mathbb{R}^n\ni v\to l(v,v_1,\dots,v_k)\in\mathbb{R}^n$$

belongs to the Lie algebra $\mathcal{G}$ (see, e.g., [4]). In components, elements of $\mathcal{G}_k$ are tensors $T^i_{jm_1\dots m_k}$ of type $\begin{pmatrix}1\\k+1\end{pmatrix}$, which are symmetric over all lower indices and, for all fixed values of $m_1,\dots,m_k$, $T^i_{jm_1\dots m_k}$ belongs to the Lie algebra $\mathcal{G}$.

The following textbook example illustrates the relation between the Cartan prolongation of the space of infinitesimal isometries and the rigidity of the structures.

Example 3. Let $\mathbb{R}^{2n}$ have Cartesian coordinates (x^i) and consider the $2n\times 2n$ unity matrix $I=I_{2n}$ and the antisymmetric matrix $J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$. The unity matrix defines the standard Euclidean metric $G=\sum_{i=1}^{2n}(dx^i)^2$, whilst the matrix J defines a symplectic structure (in Darboux coordinates) $\omega=\sum_{i=1}^{n}dx^i\wedge dx^{i+n}$.

Let $\mathbf{K}=K^i(x)\partial_i$ be a Killing vector field preserving the metric G (an infinitesimal isometry):

$$\mathcal{L}_{\mathbf{K}}G=0,\quad \text{i.e.}\quad \frac{\partial K^i(x)}{\partial x^j}+\frac{\partial K^j(x)}{\partial x^i}=0, \tag{5}$$

where $\mathcal{L}$ is the Lie derivative. Similarly, let $\mathbf{L}$ be a vector field preserving the symplectic structure ω:

$$\mathcal{L}_{\mathbf{L}}\omega=0,\quad \text{i.e.}\quad \frac{\partial L^m(x)}{\partial x^j}J_{mi}+J_{jm}\frac{\partial L^m(x)}{\partial x^i}=0. \tag{6}$$

If we differentiate Eq. (5) by any coordinate x^k, we come to

$$T^i_{kj}+T^j_{ki}=0,\qquad \text{where}\qquad T^i_{kj}=\frac{\partial^2 K^i}{\partial x^k\partial x^j}. \tag{7}$$

The tensor T^i_{kj} is symmetric with respect to the lower indices k and j, and antisymmetric in indices j and i by (7). Hence this tensor vanishes:

$$T^i_{kj}=-T^j_{ki}=-T^j_{ik}=T^k_{ij}=T^k_{ji}=-T^i_{jk}=-T^i_{kj}. \tag{8}$$

Since $T^i_{kj}\equiv 0$, we see that $K^i=c^i+B^i_jx^j$ and so all infinitesimal isometries of the Euclidean metric are translations and infinitesimal rotations. Now notice that Eq. (7) reads that the tensor T^i_{kj} belongs to $\mathfrak{so}_1(n)$, the first Cartan prolongation of the special orthogonal algebra $\mathfrak{so}(n)$, and Eq. (8) reads that the first Cartan prolongation of the algebra $\mathfrak{so}(n)$ vanishes. This algebraic fact explains the rigidity of Riemannian geometry.

In the symplectic case, Eq. (6) can be rewritten as

$$\frac{\partial L_i}{\partial x^j}=\frac{\partial L_j}{\partial x^i},\qquad \text{where}\quad L_i=L^mJ_{mi},\qquad \text{since}\quad J_{im}=-J_{mi}. \tag{9}$$

We see that this equation, contrary to Eq. (7), has an infinite dimensional space of solutions. Every Hamiltonian function Φ defines $L_i=\partial_i\Phi(x)$, which is a solution of Eq. (9) (respectively every Hamiltonian vector field $L^i\partial_i=J^{ij}\partial_j\Phi\partial_i$ is a solution of Eq. (6)). In other words, all Cartan prolongations $\mathfrak{sp}_k(n)$ of the symplectic Lie algebra $\mathfrak{sp}(n)$ are non-trivial. An arbitrary rank $k+2$ symmetric tensor $L_{m_1\cdots m_kjr}$ defines a tensor $L^i_{m_1\cdots m_kj}=L_{m_1\cdots m_kjr}J^{ri}$ belonging to the k^{th} prolongation of the symplectic algebra.

Because of the subtleties with the symmetry of odd tensors, this example suggests that the distinguishing feature between odd Riemannian and odd symplectic structure should be the rigidity of the structures described in terms of the Cartan prolongation of the corresponding Lie algebras.

Example 4. Consider the superspace $\mathbb{R}^{n|n}$ together with the odd symmetric tensor field $\mathbf{E}_{odd\,symp.}$ defined by the symmetric matrix S in Eq. (2), and the odd antisymmetric tensor field $\mathbf{E}_{odd\,riem.}$ defined by the antisymmetric matrix G in the same Eq. (2).

One can show in a similar way to Example 3 that the space of vector fields preserving the symplectic structure $\mathbf{E}_{odd\,symp.}$ is infinite dimensional, and the space of vector fields preserving the Riemannian structure is finite dimensional. Indeed, the first Cartan prolongation of the Lie algebra of vector fields preserving $\mathbf{E}_{odd\,riem.}$ vanishes. Namely, if we denote by $\mathbf{K}'$ a vector field which preserves the tensor field $\mathbf{E}_{odd\,riem.}$ (compare with Eq. (5)), then we come to the equation $T'^A_{CB}=-T'^B_{CA}(-1)^{p(B)p(A)}$, for $T'^A_{CB}=\dfrac{\partial^2K'^A}{\partial z^Cz^B}$.

Compare it with Eq. (7). This equation reads that the tensor T'^{A}_{BC} belongs to the first Cartan prolongation of the Lie algebra of vector fields preserving the odd tensor field $\mathbf{E}_{odd\,riem.}$, and it vanishes identically in the same way as its counterpart in Eq. (8).

For the odd symplectic structure $\mathbf{E}_{odd\,symp.}$ the conclusions are analogous to those for the symplectic structure in Example 3. Every rank $k+2$ symmetric tensor defines an element in the k^{th} Cartan prolongation of the Lie algebra of vector fields preserving $\mathbf{E}_{odd\,symp.}$ (the odd symplectic structure). Equivalently, every Hamiltonian function $\Phi(x)$ defines a Hamiltonian vector field preserving the odd symplectic structure.

Remark 1. We would like to note article [5], in which vector fields which simultaneously preserve both even and odd non-degenerate Poisson brackets were studied. It was shown in this paper that the space of these vector fields is finite dimensional. This fact was deduced from considerations, which implicitly involved the calculation of Cartan prolongations (the vanishing of the first Cartan prolongation of the Lie algebra of vector fields preserving both even and odd brackets).

Odd Second Order Operators and Odd Poisson Structure. Return now to an odd symmetric tensor field $\mathbf{E} = E^{AB}(z)\partial_B \otimes \partial_A$ defined on a supermanifold M. To begin, we will briefly recall some of the results from [6]. Denote by $\mathcal{F}_{\mathbf{E}}$ the class of odd second order self-adjoint differential operators with the principal symbol $\mathbf{E}$ acting on half-densities on this supermanifold M

$$\mathcal{F}_{\mathbf{E}} \in \Delta\colon \quad \Delta = \frac{1}{2}\left(E^{AB}(z)\partial_B\partial_A + \partial_B E^{BA}(z)\partial_A + U(z)\right), \tag{10}$$

where $U(z)$ is an odd function on M, $p(U)$=1. If $\mathbf{s} = s(z)\sqrt{Dz}$ is a half-density, then

$$\Delta\mathbf{s} = \frac{1}{2}\left(\partial_B\left(E^{BA}\partial_A s(z)\right) + U(z)s(z)\right)\sqrt{Dz}.$$

Any two operators in $\mathcal{F}_{\mathbf{E}}$ differ by an odd scalar function (see for detail [6]).

The term $U = U(z)$ is called the potential field, and transforms under a change of local coordinates in the following way:

$$U' = U + \frac{1}{2}\partial_{A'}\left(E^{A'B'}\partial_{B'}\log J\right) + \frac{1}{4}\left(\partial_{A'}\log J E^{A'B'}\partial_{B'}\log J\right), \tag{11}$$

where J is the Berezinian (superdeterminant) of the Jacobian of the coordinate change. The potential field acts as a second order compensation field (a second order connection) on the manifold M; as a first order connection compensates the action of diffeomorphisms on the first derivatives, the potential field compensates the action on the second derivatives.

We now consider the operator Δ^2. Since Δ is an odd self-adjoint operator of order 2, Δ^2 is an even anti-self-adjoint operator of order equal to either 3, 1 or else $\Delta^2 = 0$. The condition that $\mathbf{E}$ defines a Poisson structure and the relations between this structure and the class $\mathcal{F}_{\mathbf{E}}$ of operators can be summarised in the following statement.

Proposition 1 [6]. Let Δ be an arbitrary operator in the class $\mathcal{F}_{\mathbf{E}}$. Then the odd symmetric tensor field $\mathbf{E}$ defines an odd Poisson structure on the manifold M, i.e. it obeys the Jacobi identity, if and only if the order of the operator Δ^2 is equal to 1 or else $\Delta^2 = 0$. In this case the operator Δ^2 defines a vector field $\mathbf{X} = \mathbf{X}_\Delta$ such that

$$\Delta^2 = \mathcal{L}_{\mathbf{X}}, \tag{12}$$

where $\mathcal{L}_{\mathbf{X}}$ is the Lie derivative along the vector field $\mathbf{X}$. The vector field $\mathbf{X} = \mathbf{X}_\Delta$ is called the modular vector field of the operator Δ, and preserves the Poisson structure. If Δ' is another arbitrary operator from the class $\mathcal{F}_{\mathbf{E}}$, that is, $\Delta' = \Delta + F$, then $\mathbf{X}_{\Delta'} = \mathbf{X}_\Delta + D_F$, where D_F is the even Hamiltonian vector field corresponding to the odd function F. The corresponding equivalence class of the modular vector field (in Lichnerowicz–Poisson cohomology) is called the modular class of the odd Poisson manifold.

If $\mathbf{E}$ defines an odd Poisson structure on M, then the modular vector field $\mathbf{X} = \mathbf{X}_\Delta$ of operator (10) has the following local appearance:

$$\mathbf{X} = \frac{1}{2}\partial_C \left(E^{CD}\partial_D \partial_B E^{BA}\right)\partial_A + (-1)^{p(A)} E^{AB}\partial_B U \partial_A . \tag{13}$$

Assume now that the tensor field $\mathbf{E}$ defines an odd non-degenerate Poisson structure (an odd symplectic structure) on the supermanifold M. In Darboux coordinates (x^i, θ_j) (see Example 1), one may naively define the canonical operator Δ on half-densities by

$$\Delta \mathbf{s} = \frac{\partial^2 s(x,\theta)}{\partial x^i \partial \theta_i}\sqrt{D(x,\theta)}. \tag{14}$$

What is remarkable is that this local expression defines an operator globally on M [7]; in arbitrary Darboux coordinates it has the same appearance. In other words, the vanishing of the potential U in one set of Darboux coordinates implies that it vanishes in arbitrary Darboux coordinates. We see that if $\mathbf{E}$ defines a non-degenerate odd Poisson structure, then the class $\mathcal{F}_\mathbf{E}$ possesses a distinguished operator, the odd canonical operator (14) defined by the condition

$$U = 0 \quad \text{in Darboux coordinates}. \tag{15}$$

The expression for the potential of the odd canonical operator was calculated in arbitrary coordinates in [8]. It has the appearance

$$U(z) = \frac{1}{4}\partial_B \partial_A E^{AB}(z) - (-1)^{p(B)(p(D)+1)}\frac{1}{12}\partial_A E^{BC}(z) E_{CD}(z)\partial_B E^{DA}(z), \tag{16}$$

where E_{AB} is the inverse tensor to E^{AB}.

Proposition 2. For an odd symplectic supermanifold there exists a unique potential field U defining the odd canonical operator (14). This potential field vanishes in arbitrary Darboux coordinates. In arbitrary local coordinates it is given by expression (16). The potential U acts as a second order compensating field and transforms under a change of coordinates according to (11).

This proposition can be considered as a far analogue to the Riemannian case of the unique first order compensating field, i.e. the Levi-Civita connection.

Notice that the modular vector field (12) of the odd canonical operator vanishes. (In particular, this means that the modular class of an odd symplectic manifold vanishes.) The vanishing of the modular vector field of the canonical operator means that the potential U in (16) is a solution of the first order differential equations

$$\frac{1}{2}\partial_C \left(E^{CD}\partial_D \partial_B E^{BA}\right) + (-1)^{p(A)} E^{AB}\partial_B U = 0,$$

which follow from Eq. (13). The solution of these equations is unique up to an odd constant. Condition (15) implies that this odd constant vanishes for the canonical operator (14) (see also [9]).

Remark 2. We would like to note that we currently have no conceptually clear way of deriving formula (16). On the other hand, the vector field (13) has some mysterious properties, which may be part of a calculus for odd Poisson manifolds and in particular for odd symplectic geometry. We think that understanding these properties will elucidate the geometrical structure of expression (16). This is a work in progress.

We are grateful to Th. Voronov for encouraging discussions.

Received 06.07.2016

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