

ELECTROMAGNETIC CASIMIR DENSITIES FOR A PLATE
IN ANTI-DE SITTER SPACETIME

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We investigate the vacuum expectation values (VEV) of the electric field squared and of the energy-momentum tensor for the electromagnetic field in anti-de Sitter (AdS) spacetime induced by a plate parallel to the AdS boundary. On the plate the field obeys the boundary condition that generalizes the perfect conductor boundary condition for an arbitrary number of spatial dimensions. We show that the plate-induced contributions in the VEV of the electric field squared and the vacuum energy density are negative, whereas the normal stress is positive. The VEV vanish on the AdS boundary.

Keywords: anti-de Sitter space, Casimir effect, quantum vacuum.

1. Introduction. Much of early interest to anti-de Sitter (AdS) spacetime was motivated by the questions of principle nature related to the quantization of fields propagating on curved backgrounds. The appearance of AdS/CFT correspondence and braneworld models of Randall–Sundrum type has revived interest in this subject considerably. AdS/CFT correspondence [1] relates string theories or supergravity in AdS bulk with a conformal field theory living on its boundary. It has many interesting consequences and provides a powerful tool for the investigation of gauge field theories. The braneworld scenario [2] offers a new perspective on the hierarchy problem between the gravitational and electroweak mass scales. The main idea to resolve the large hierarchy is that the small coupling of four-dimensional gravity is generated by the large physical volume of extra dimensions. Braneworlds naturally appear in string/M-theory context and present intriguing possibilities to solve or to address from a different point of view various problems in particle physics and cosmology.

An inherent feature of all these models is that the boundary conditions on the fields should be specified in order to completely determine the dynamics. The boundary conditions, imposed on the field operator, modify the spectrum of the vacuum fluctuations of a quantum field. As a consequence, the vacuum expectation values (VEV) of physical observables are shifted. This general phenomena is known as the Casimir effect (for a review see [3]). In braneworlds, the boundary conditions imposed on the bulk fields will give Casimir-type contributions to vacuum energy and to vacuum forces acting on the branes. The latter provide a natural mechanism for stabilizing the interbrane distance in Randall–Sundrum-type models.

In the present paper we consider the influence of the plate parallel to AdS boundary on the properties of the electromagnetic field. On the plate the boundary condition is imposed that is a generalization of the perfect conductor boundary condition in 4-dimensional spacetime. The two-point function for electromagnetic field in the boundary-free AdS spacetime

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is investigated in [4]. The electromagnetic Casimir densities in de Sitter spacetime for flat boundaries have been considered in [5, 6]. The electromagnetic two-point functions and the Casimir effect in background of more general Friedmann–Robertson–Walker cosmologies are discussed in [7].

Outline of the paper is as follows. In next Section we describe the geometry of the problem and present the mode functions. The VEV of the electric field squared and of the energy-momentum tensor in the region between the plate and AdS boundary are investigated in Sections 3 and 4 respectively.

2. Background Geometry and the Modes. We consider the quantum electromagnetic field in background of $(D+1)$ -dimensional AdS spacetime. The latter will be described in the Poincaré coordinates with the line element

$$ds^2 = g_{ik} dx^i dx^k = e^{-2y/\alpha} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \quad (1)$$

where Latin and Greek indices run over the values $0, 1, \dots, D$ and $0, 1, \dots, D-1$ respectively, $\eta_{\mu\nu} = \text{diag}(1, -1, \dots, -1)$ and α is the curvature radius. For the corresponding curvature scalar one has $R = -D(D+1)/\alpha^2$.

In addition to the coordinate y , we will use the coordinate $z = \alpha e^{y/\alpha}$, $0 \leq z < \infty$, in terms of which the line element takes a conformally-flat form $ds^2 = (\alpha/z)^2 \eta_{ik} dx^i dx^k$ with $x^D = z$. Hypersurfaces $z = 0$ and $z = \infty$ correspond to AdS boundary and horizon respectively.

We assume the presence of a boundary located at $z = z_0$, on which the electromagnetic field obeys the boundary condition

$$n^{\nu_1} {}^*F_{\nu_1 \dots \nu_{D-1}} = 0, \quad (2)$$

where n^{ν_1} is the normal vector to the boundary, ${}^*F_{\nu_1 \dots \nu_{D-1}}$ is the dual of the field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and A_μ is the vector potential. This condition generalizes the perfectly conducting boundary condition in standard $D = 3$ electrodynamics.

We are interested in the VEV of the electric field squared and of the energy-momentum tensor for electromagnetic field in the region $0 \leq z \leq z_0$. For evaluation of the VEV a complete set of electromagnetic modes, obeying condition (2), is required. This set can be found in a way similar to that was used in [8] for the region $z_0 \leq z < \infty$. In the gauge $A_D = 0$, $\nabla_i A^i = 0$ the corresponding eigenfunctions, regular on AdS boundary, have the form

$$A_{(\sigma\lambda\mathbf{k})l} = C \varepsilon_{(\sigma)l} z^{D/2-1} J_{D/2-1}(\lambda z) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}, \quad (3)$$

where $J_\nu(x)$ is the Bessel function, $\mathbf{x} = (x^1, x^2, \dots, x^{D-1})$, $\mathbf{k} = (k^1, k^2, \dots, k^{D-1})$, $\omega = \sqrt{\lambda^2 + k^2}$. The index $\sigma = 1, \dots, D-1$ corresponds to different polarizations. The polarization vectors are normalized by relation $\eta^{\mu\nu} \varepsilon_{(\sigma')\mu} \varepsilon_{(\sigma)\nu} = -\delta_{\sigma\sigma'}$. From the gauge conditions we have $\varepsilon_{(\sigma)D} = 0$ and $k^\mu \varepsilon_{(\sigma)\mu} = 0$. Combining these relations, one can see that

$$\sum_{\sigma=1}^{D-1} \varepsilon_{(\sigma)\mu} \varepsilon_{(\sigma)\nu} = -\eta_{\mu\nu} + k_\mu k_\nu / \lambda^2. \quad (4)$$

From the boundary condition (2) at $z = z_0$ it follows that the eigenvalues of λ are the roots of the equation

$$J_{D/2-1}(\lambda z_0) = 0. \quad (5)$$

We denote the zeros of the function $J_{D/2-1}(x)$ by j_n with $n = 1, 2, \dots$, $0 < j_n < j_{n+1}$. Then, for the eigenvalues of the λ , we get $\lambda = j_n / z_0$. The coefficient C in (3) is found from the normalization condition and is given by

$$C^2 = \frac{2(2\pi)^{2-D} \alpha^{3-D}}{z_0^2 \omega J_{D/2-1}'^2(j_n)}. \quad (6)$$

Note that one has $J_{D/2-1}'(j_n) = -J_{D/2}(j_n)$.

3. VEV of the Electric Field Squared. In this section we investigate the VEV of the electric field squared $\langle 0|E^2|0\rangle \equiv \langle E^2\rangle$, where $|0\rangle$ stands for the vacuum state. This VEV is evaluated by using the mode-sum formula

$$\langle E^2\rangle = g^{00}g^{il} \sum_{\sigma=1}^{D-1} \int d\mathbf{k} \sum_{n=1}^{\infty} F_{(\sigma\lambda\mathbf{k})0i}(x)F_{(\sigma\lambda\mathbf{k})0i}^*(x), \quad (7)$$

where $F_{(\sigma\lambda\mathbf{k})il}(x)$ is the field tensor corresponding to the vector potential (3) and the star means the complex conjugate. The expression in the right hand side of (7) diverges. In order to have a finite expression, we assume that some cutoff function is introduced without writing it explicitly. The special form of that function will not be important for the further discussion.

The expression (7) contains the summation over the zeros of the Bessel function $J_{D/2-1}(x)$. These zeros are given implicitly and this expression is not convenient for the numerical evaluation. For a function $f(u)$ analytic in the right half plane $\text{Re } u > 0$, we use the generalized Abel–Plana formula [9]

$$\sum_{n=1}^{\infty} \frac{f(j_n)}{j_n J_{D/2-1}^2(j_n)} = \frac{1}{2} \int_0^{\infty} du f(u) + \frac{1}{2\pi} \int_0^{\infty} du \frac{K_{D/2-1}(u)}{I_{D/2-1}(u)} \left[e^{-\pi i D/2} f(iu) + e^{\pi i D/2} f(-iu) \right], \quad (8)$$

where $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions. It can be shown that the contribution of the first integral in (8) to $\langle E^2\rangle$ corresponds to the VEV of the electric field squared in the geometry without the plate $z = z_0$. The latter will be denoted by $\langle E^2\rangle_0$. As a consequence, we get the following decomposition

$$\langle E^2\rangle = \langle E^2\rangle_0 + \langle E^2\rangle_b, \quad (9)$$

where $\langle E^2\rangle_b$ is induced by the plate. For points outside of the plate, all the divergences are contained in the boundary-free part only and renormalization is reduced to the renormalization of the latter. From the maximal symmetry of AdS bulk we expect that renormalized VEV $\langle E^2\rangle_0$ does not depend on the spacetime point.

For the boundary-induced contribution we find the following expression

$$\langle E^2\rangle_b = -\frac{(D-1)(z/z_0)^{D+2}}{(4\pi)^{D/2-1} \Gamma(D/2+1) \alpha^{D+1}} \int_0^{\infty} dx x^{D+1} = \frac{K_{D/2-1}(x)}{I_{D/2-1}(x)} \left[I_{D/2-2}^2(xz/z_0) + 2I_{D/2-1}^2(xz/z_0) \right]. \quad (10)$$

This part is always negative. For the special case $D = 3$ the VEV reduces to

$$\langle E^2\rangle_b = \frac{(z/z_0)^4}{4\pi\alpha^4} \left[\frac{\pi^2}{15} - \int_0^{\infty} dx \frac{x^3 \cosh(xz/z_0)}{e^x - 1} \right]. \quad (11)$$

The Minkowskian limit corresponds to $\alpha \rightarrow \infty$ for a fixed value of the coordinate y . By taking into account the relation between the coordinates y and x , we find $z \approx \alpha + y$ and $z/z_0 \approx 1 + (y - y_0)/\alpha$. The dominant contribution to the integral in (10) comes from large values of the integration variable. By using the asymptotic expressions for the modified Bessel functions for large arguments, we find $\langle E^2\rangle_b \rightarrow \langle E^2\rangle_b^{(M)}$, where

$$\langle E^2\rangle_b^{(M)} = -\frac{3(D-1)\Gamma((D+1)/2)}{2(4\pi)^{(D-1)/2}(y_0 - y)^{D+1}} \quad (12)$$

is the VEV induced by a plate in Minkowski spacetime located at $y = y_0$.

The VEV (10) diverges on the plate $z = z_0$. For points near the plate, assuming that $1 - z/z_0 \ll 1$, the leading term in the asymptotic expansion over the distance from the plate is

given by (12). On AdS boundary, corresponding to $z = 0$, the boundary-induced VEV $\langle E^2 \rangle_b$ vanishes. The leading term is given by the expression

$$\langle E^2 \rangle_b \approx -\frac{2^{6-2D}\pi^{1-D/2}(D-1)(z/z_0)^{2D-2}}{\Gamma(D/2+1)\Gamma^2(D/2-1)\alpha^{D+1}} \int_0^\infty dx x^{2D-3} \frac{K_{D/2-1}(x)}{I_{D/2-1}(x)} \quad (13)$$

for $D \geq 4$ and by $\langle E^2 \rangle_b \approx \frac{-2\pi^5\alpha^{-4}(z/z_0)^6}{63}$ for $D = 3$.

4. VEV of the Energy-Momentum Tensor. For the evaluation of the VEV of the energy-momentum tensor we employ the mode-sum formula

$$\langle T_i^k \rangle = -\frac{1}{4\pi} \sum_{\sigma=1}^{D-1} \int d\mathbf{k} \sum_{n=1}^{\infty} \left[F_{(\sigma\lambda\mathbf{k})il}(x) F_{(\sigma\lambda\mathbf{k})}^{kl}(x) - \frac{1}{4} \delta_i^k F_{(\sigma\lambda\mathbf{k})lm}(x) F_{(\sigma\lambda\mathbf{k})}^{lm}(x) \right]. \quad (14)$$

Similar to the case of the field squared, by applying to the series over n the summation formula (8), the VEV is written in the decomposed form

$$\langle T_i^k \rangle = \langle T_i^k \rangle_0 + \langle T_i^k \rangle_b. \quad (15)$$

For points outside the plate the renormalization is required for the boundary-free part only. Because of the maximal symmetry of the background geometry, the latter is proportional to the metric tensor: $\langle T_i^k \rangle_0 = \text{const} \cdot \delta_i^k$. The boundary-induced contribution in the VEV of the vacuum energy-momentum tensor is diagonal. For the corresponding components we obtain (no summation over l)

$$\langle T_l^l \rangle_b = -\frac{(D-1)(z/z_0)^{D+2}}{(4\pi)^{D/2}\Gamma(D/2+1)\alpha^{D+1}} \int_0^\infty dx x^{D+1} \frac{K_{D/2-1}(x)}{I_{D/2-1}(x)} G_{D/2-1}^{(l)}(xz/z_0), \quad (16)$$

where

$$G_v^{(l)}(u) = vI_{v-1}^2(u) + (v-1)I_v^2(u), \quad l = 0, 1, \dots, D-1; \quad G_v^{(D)}(u) = (v+1)[I_v^2(u) - I_{v-1}^2(u)].$$

It can be checked that this VEV obeys the covariant conservation equation $\nabla_k \langle T_i^k \rangle_b = 0$. In the geometry under consideration this is reduced to a single equation

$$z^{D+1} \partial_z (z^{-D} \langle T_D^D \rangle_b) + D \langle T_0^0 \rangle_b = 0. \quad (17)$$

The plate-induced contribution to the energy density is negative, $\langle T_0^0 \rangle_b < 0$, whereas the normal stress is positive, $\langle T_D^D \rangle_b > 0$, i.e. the corresponding vacuum pressure is negative. The vacuum stresses parallel to the plate are equal to the energy density. The latter property is a direct consequence of the Lorentz invariance along the directions parallel to the plates.

In the special case $D = 3$ one gets

$$\langle T_D^D \rangle_b = \frac{\pi^2(z/z_0)^4}{240\alpha^4}, \quad \langle T_l^l \rangle_b = -\frac{1}{3} \langle T_D^D \rangle_b \quad (18)$$

for $l = 0, 1, \dots, D-1$. In this special case, the electromagnetic field is conformally invariant and the boundary-induced contribution is expressed in terms of the corresponding VEV in the region between two conducting plates in Minkowski bulk by the standard conformal relation.

For points near the plate and for $D \geq 4$, the dominant contribution to (16) comes from large values of x and to the leading order we find

$$\langle T_l^l \rangle_b \approx -\frac{(D-3)(D-1)\Gamma((D+1)/2)}{2(4\pi)^{(D+1)/2}(y_0-y)^{D+1}}, \quad \langle T_D^D \rangle_b \approx \frac{y-y_0}{\alpha} \langle T_0^0 \rangle_b \quad (19)$$

for $l = 0, 1, \dots, D-1$. The leading term in the components with $l = 0, 1, \dots, D-1$ coincides with the corresponding result for a plate in the Minkowski bulk. Near AdS boundary, $z \rightarrow 0$, the boundary-induced VEV vanish with the leading terms

$$\langle T_D^D \rangle_b \approx \frac{2^{4-2D} (D-1) (z/z_0)^{2D-2}}{\pi^{D/2} \Gamma(D/2) \Gamma^2(D/2-1) \alpha^{D+1}} \int_0^\infty dx x^{2D-3} \frac{K_{D/2-1}(x)}{I_{D/2-1}(x)} \quad (20)$$

and $\langle T_l^l \rangle_b \approx (2/D-1) \langle T_D^D \rangle_b$ for $l = 0, 1, \dots, D-1$.

5. Conclusion. We have discussed the effects of a boundary in AdS spacetime on the properties of the electromagnetic vacuum in an arbitrary number of spatial dimensions. The boundary is parallel to AdS boundary and on it the electromagnetic field obeys the condition (2) that corresponds to the standard perfect conductor boundary condition for $D = 3$. The plate-induced contribution to the VEV of the electric field squared in the region $0 \leq z \leq z_0$ is given by (10). It is negative everywhere, vanishes on AdS boundary and diverges on the plate. For the plate-induced part in the vacuum energy-momentum tensor we have derived the expression (16). The corresponding energy density and stresses along the directions parallel to the plate are negative, whereas the normal stress is positive.

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REFERENCES

1. **Aharony O., Gubser S.S., Maldacena J., Ooguri H., Oz Y.** Large N Field Theories, String Theory and Gravity. // *Phys. Rep.*, 2000, v. 323, p. 183–386.
2. **Maartens R., Koyama K.** Brane-World Gravity. // *Living Rev. Relativity*, 2010, v. 13, № 5, p. 1–124.
3. **Bordag M., Klimchitskaya G.L., Mohideen U., Mostepanenko V.M.** *Advances in the Casimir Effect.* Oxford: Oxford University Press, 2009, 768 p.
4. **Allen B., Jacobson T.** Vector Two-Point Functions in Maximally Symmetric Spaces. // *Commun. Math. Phys.*, 1986, v. 103, № 4, p. 669–692.
5. **Saharian A.A., Kotanjyan A.S., Nersisyan H.A.** Electromagnetic Two-Point Functions and Casimir Densities for a Conducting Plate in de Sitter Spacetime. // *Phys. Lett. B*, 2014, v. 728, p. 141–147.
6. **Kotanjyan A.S., Saharian A.A., Nersisyan H.A.** Electromagnetic Casimir Effect for Conducting Plates in de Sitter Spacetime. // *Phys. Scr.*, 2015, v. 90, p. 065304.
7. **Bellucci S., Saharian A.A.** Electromagnetic Two-Point Functions and the Casimir Effect in Friedmann–Robertson–Walker Cosmologies. // *Phys. Rev. D*, 2013, v. 88, p. 064034.
8. **Kotanjyan A.S., Saharian A.A., Sargsyan H.G., Simonyan D.H.** Quantum Effects in Anti-de Sitter Spacetime for the Electromagnetic Field. // *Proceedings of Science, PoS*, 2015, v. 021, p. 1–13.
9. **Saharian A.A.** *The Generalized Abel–Plana Formula with Applications to Bessel Functions and Casimir Effect.* Yer.: YSU Press, 2008, 138 p.