PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2017, **51**(1), p. 18–21

M a t h e m a t i c s

AN EXAMPLE OF DOUBLE FOURIER–HAAR SERIES WITH A NONREGULAR SUBSERIES

K. A. KERYAN *

Chair of Mathematical Analysis and Function Theory YSU, Armenia
College of Science and Engineering AUA, Armenia

We bring an example of integrable function on $[0,1]^2$, so that the double Fourier–Haar series has a subseries, whose majorant of partial sums does not belong to $L^{1,\infty}$.

MSC2010: Primary 42C10; Secondary 26A39.

***Keywords*:** double Haar series, majorant, square function, A-integral.

Introduction. We construct an example of double Fourier–Haar series with a subseries having nonregular distribution of the majorant. In order to formulate the results recall the definition of Haar system on $[0,1]$ as follows.

The first Haar function is: $\chi_1(x)=1$. Then for $n=2^k+i$, where $i=1,2,...,2^k$, $k=0,1,2,...$, denote

$$\chi_n(x)=\chi_i^{(k)}(x)=\begin{cases} 2^{\frac{k}{2}}, & \text{if } \dfrac{i-1}{2^k}\le x<\dfrac{2i-1}{2^{k+1}},\\ 2^{-\frac{k}{2}}, & \text{if } \dfrac{2i-1}{2^{k+1}}\le x<\dfrac{i}{2^k},\\ 0, & \text{if } x\notin\left[\dfrac{i-1}{2^k},\dfrac{i}{2^k}\right). \end{cases} \tag{1}$$

For $n=2^k+i$ denote $\{n\}=\operatorname{supp}\chi_n=\left[\dfrac{i-1}{2^k},\dfrac{i}{2^k}\right)$.

We say a function f is A-integrable on a set G, if $\lim\limits_{\lambda\to\infty}\lambda\cdot\operatorname{mes}\{\mathbf{x}\in G:|f(\mathbf{x})|>\lambda\}=0$ and the limit

$$\lim_{\lambda\to\infty}\int_G[f(\mathbf{x})]_\lambda d\mathbf{x}=:(A)\int_G f(\mathbf{x})d\mathbf{x} \tag{2}$$

exists, where

$$[f(\mathbf{x})]_\lambda=\begin{cases} f(\mathbf{x}), & \text{if } |f(\mathbf{x})|\le\lambda,\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

Preliminary Theorems. The following theorem was proved in [1].

T h e o r e m A 1. Let the series

$$\sum_{n=1}^{\infty}a_n\chi_n(x) \tag{4}$$

* E-mail: karenkeryan@ysu.am, kkeryan@aua.am

be Fourier–Haar series of some integrable function and ε_n, $n \in \mathbb{N}$, be a bounded sequence. Then the series $\sum_{n=1}^{\infty} \varepsilon_n a_n \chi_n(x)$ is a Fourier–Haar series of an A-integrable function, i.e. there exists an A-integrable function f_ε, so that for any n we have

$$\varepsilon_n a_n = (A)\int_0^1 f_\varepsilon(x)\chi_n(x)dx.$$

Theorem A1 in case $\varepsilon_n = 0$ or 1 was stated by L.A. Balashov [2].

It can also be proved applying the following theorems from [1].

Theorem A 2. The conditions

$$\lim_{\lambda\to\infty} \lambda \cdot \operatorname{mes}\left\{ x \in [0,1] : \left\{ \sum_{n=1}^{\infty} a_n^2 \chi_n^2(x) \right\}^{1/2} > \lambda \right\} = 0 \tag{5}$$

and

$$\lim_{\lambda\to\infty} \lambda \cdot \operatorname{mes}\left\{ x \in [0,1] : S^*(x) := \sup_N \left| \sum_{n=1}^{N} a_n \chi_n(x) \right| > \lambda \right\} = 0 \tag{6}$$

are equivalent to each other.

Theorem A 3. If the series (4) satisfies the condition (6), then it is a Fourier–Haar series in the sense of A-integrability.

The following theorem is proved in [3].

Theorem A 4. The condition (5) is satisfied if and only if for any bounded sequence $\left\{\varepsilon_n\right\}_{n=1}^{\infty}$ the series $\sum_{n=1}^{\infty} \varepsilon_n a_n \chi_n(x)$ is a Fourier–Haar series in the sense of A-integrability.

In particular, it was proved in [1], that if the function $S^*(x,y) = \sup_k \left| \sum_{m,n=1}^{2^k} a_{mn}\chi_m(x)\chi_n(y) \right|$ satisfies a condition analogous to (6), then the series $\sum_{m,n=1}^{\infty} a_{mn}\chi_m(x)\chi_n(y)$ is a Fourier–Haar series of an A-integrable function in the sense of A-integrability. Nevertheless, in contrast to the one dimensional case, the analogue of Theorem A1 does not hold for the double Fourier–Haar series.

Main Result.

Theorem 1. There exists a series $\sum_{m,n=1}^{\infty} a_{m,n}\chi_m(x)\chi_n(y)$, which is a Fourier–Haar series of Lebesgue integrable function on $[0,1]^2$ and a sequence $\varepsilon_{mn} = 0,\ 1,\ (m,n) \in \mathbb{N}^2$, such that

$$\limsup_{\lambda\to\infty} \lambda \cdot \operatorname{mes}\{\mathbf{x} \in [0,1]^2 : S^*(x,y) > \lambda\} = \infty, \tag{7}$$

where $S^*(x,y) = \sup_k \left| \sum_{m,n=1}^{2^k} \varepsilon_{mn} a_{mn}\chi_m(x)\chi_n(y) \right|$.

Proof. It is known and easy to verify that the 2^kth square partial sums of the series

$$\sum_{m,n=1}^{\infty} a_{mn}\chi_m(x)\chi_n(y), \quad \text{where } a_{mn} = a_m a_n \text{ and } a_m = \chi_m(+0), \tag{8}$$

satisfy the equality (see [4])

$$\sigma_k(x,y) := \sum_{m,n=1}^{2^k} a_{mn}\chi_m(x)\chi_n(y) = \sum_{m=1}^{2^k} a_m\chi_m(x) \sum_{n=1}^{2^k} a_n\chi_n(y) = 2^{2k}\mathbb{1}_{[0,2^{-k}]^2}(x,y). \tag{9}$$

For any positive sequence $\alpha_j,\ j\in\mathbb{N}$, satisfying

$$\sum_{j=1}^{\infty} \alpha_j < \infty, \tag{10}$$

and for any increasing sequence $k_j\in\mathbb{N}$, satisfying

$$k_{j+1} > 3\cdot k_j, \tag{11}$$

the series

$$\sum_{j=1}^{\infty} \alpha_j(\sigma_{k_{2j+1}}(x,y) - \sigma_{k_{2j}}(x,y)) =: \sum_{j=1}^{\infty} \alpha_j\Psi_j(x,y) \tag{12}$$

is a Fourier–Haar series of a Lebesgue integrable function.

Indeed, it follows from (9), that $\|\Psi_j\|_1 \le 2$. Therefore, applying (10) and taking into account the orthogonality of the functions $\Psi_j,\ j\in\mathbb{N}$, one can obtain that the series (12) is a Fourier–Haar series of its sum, which is an integrable function.

Note, that if $m = 2^p + 1$, then $a_m = 2^{\frac{p}{2}}$. Hence, the condition (11) guarantees for the sum (see also (1))

$$g_j(x,y) := \sum_{p=k_{2j}+1}^{3k_{2j}} 2^{2k_{2j}}\chi_1^{(p)}(x)\chi_1^{(4k_{2j}-p)}(y) \tag{13}$$

to be a subsum of the finite sums

$$\Psi_j(x,y) = \sum_{2^{k_{2j}}<\max(m,n)\le 2^{k_{2j+1}}} a_{mn}\chi_m(x)\chi_n(y), \tag{14}$$

i.e. there exist numbers $\varepsilon_{mn} = 0$ or 1, so that

$$g_j(x,y) = \sum_{2^{k_{2j}}<\max(m,n)\le 2^{k_{2j+1}}} \varepsilon_{mn}a_{mn}\chi_m(x)\chi_n(y). \tag{15}$$

For a fixed $j\in\mathbb{N}$ consider the rectangle $\Delta_p^j = \operatorname{supp}\left(\chi_1^{(p)}\chi_1^{(4k_{2j}-p)}\right)$. Obviously $\operatorname{mes}(\Delta_p^j) = 2^{-4k_{2j}}$. It is not hard to verify that a quarter of each rectangle Δ_p^j does not intersect the other rectangles, so we have $\operatorname{mes}\left(\Delta_p^j \Big\backslash \bigcup_{p'\ne p}\Delta_{p'}^j\right) \ge \frac{1}{4}\operatorname{mes}(\Delta_p^j)$. Thus we get

$$\operatorname{mes}(F_j) \ge \frac{1}{4}\sum_{p=k_{2j}+1}^{3k_{2j}} 2^{-4k_{2j}} \ge \frac{k_{2j}}{2}\cdot 2^{-4k_{2j}}, \text{ where } F_j := \bigcup_{p=k_{2j}+1}^{3k_{2j}}\left(\Delta_p^j\Big\backslash\bigcup_{p'\ne p}\Delta_{p'}^j\right). \tag{16}$$

Obviously (see (1) and (13))

$$\max_{k_{2j}+1\le p\le 3k_{2j}}\left|2^{2k_{2j}}\chi_1^{(p)}(x)\chi_1^{(4k_{2j}-p)}(y)\right| = 2^{4k_{2j}} \text{ for } (x,y)\in F_j. \tag{17}$$

Take $k_j = 2^{2j}$ and $\alpha_j = 2^{-j}$. We claim that the series (12) satisfies the conditions of Theorem 1. We have already noted that (12) is a Fourier–Haar series. Since (13) is a subsum of (14), for corresponding $\varepsilon_{mn} = 0,\ 1$ the series

$$\sum_{j=1}^{\infty} 2^{-j} \sum_{2^{k_{2j}}<\max(m,n)\le 2^{k_{2j+1}}} \varepsilon_{mn}a_{mn}\chi_m(x)\chi_n(y) = \sum_{j=1}^{\infty} 2^{-j}g_j(x,y) \tag{18}$$

can be considered as a subseries of (12).

Denote by $S^*(x,y)$ the majorant of 2^kth square partial sums of the series $\sum\limits_{j=1}^{\infty} 2^{-j} g_j(x,y)$, considered as a double Haar series.

It follows from (16) and (17), that

$$S^*(x,y) \geq 2^{-j-1} \cdot 2^{2^{4j+2}} \quad \text{for} \quad (x,y) \in F_j, \tag{19}$$

and

$$\operatorname{mes}(F_j) \geq 2^{4j-1} \cdot 2^{-2^{4j+2}}. \tag{20}$$

Thus, for $\lambda_j = 2^{-j-2} \cdot 2^{2^{4j+2}}$ we have

$$\lambda_j \cdot \operatorname{mes}\{(x,y) \in [0,1]^2 : S^*(x,y) > \lambda_j\} \geq 2^{3j-2}.$$

Hence,

$$\limsup_{\lambda \to \infty} \lambda \cdot \operatorname{mes}\{(x,y) \in [0,1]^2 : S^*(x,y) > \lambda\} = \infty. \qquad \square$$

It follows from (16), (17), that the series $\sum\limits_{(m,n)\in\mathbb{N}^2} a_{mn}\chi_m(x)\chi_n(y)$ does not satisfy

$$\lim_{\lambda \to \infty} \lambda \cdot \operatorname{mes}\left\{(x,y) \in [0,1]^2 : \left(\sum_{(m,n)\in\mathbb{N}^2} a_{mn}^2 \chi_m^2(x)\chi_n^2(y)\right)^{1/2} > \lambda\right\} = 0. \tag{21}$$

On the other hand, taking into account that $\sum\limits_{(m,n)\in\mathbb{N}^2} a_{mn}\chi_m(x)\chi_n(y)$ is a Fourier–Haar series, we get

$$\lim_{\lambda \to \infty} \lambda \cdot \operatorname{mes}\left\{(x,y) \in [0,1]^2 : \sup_k \left|\sum_{m,n=1}^{2^k} a_{mn}\chi_m(x)\chi_n(y)\right| > \lambda\right\} = 0. \tag{22}$$

Hence, generally speaking (21) does not follow from (22).

It is interesting to find out whether the condition (21) implies (22). If it is the case, then the following will take place: if the series $\sum\limits_{m,n=1}^{\infty} a_{mn}\chi_m(x)\chi_n(y)$ satisfies (21), then the series $\sum\limits_{m,n=1}^{\infty} \varepsilon_{mn} a_{mn}\chi_m(x)\chi_n(y)$ is a Fourier–Haar series in the sense of A-integrability for any bounded sequence ε_{mn}, $(m,n) \in \mathbb{N}^2$.

This work was supported by State Committee of Science MES RA in frame of research project SCS RA 15T–1A006.

Received 18.11.2016

REFERENCES

1. **Gevorkyan G.G.** On Uniqueness of Additive Functions of Dyadic Cubes and Series by Haar System. // J. Contemp. Math. Anal. NAS RA, 1995, v. 30, № 5, p. 2–13.
2. **Balashov L.A.** Generalized Integration of Series by Haar System. All Union School of Function Theory. Yer., 1987, p. 3.
3. **Gevorkyan G.G., Navasardyan K.A.** On Haar Series of A-Integrable Functions. // Izv. NAS RA. Matematika, 2017, v. 52, № 1, p. 47–58 (in Russian).
4. **Kashin B.S., Saakyan A.A.** Orthogonal Series. AFTs. M., 1999, 1st ed. transl. Am. Math. Soc., Providence, RI, Transl. Math. Monogr. V. 75, 1989, 449 p.