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*M a t h e m a t i c s*

# DEFICIENCY OF OUTERPLANAR GRAPHS

H. H. KHACHATRIAN *

*Chair of Discrete Mathematics and Theoretical Informatics YSU, Armenia*

An edge-coloring of a graph $G$ with colors $1,2,\ldots,t$ is an interval $t$-coloring, if all colors are used, and the colors of edges incident to each vertex of $G$ are distinct and form an interval of integers. A graph $G$ is interval colorable, if it has an interval $t$-coloring for some positive integer $t$. $def(G)$ denotes the minimum number of pendant edges that should be attached to $G$ to make it interval colorable. In this paper we study interval colorings of outerplanar graphs. In particular, we show that if $G$ is an outerplanar graph, then $def(G) \leq (|V(G)|-2)/(og(G)-2)$, where $og(G)$ is the length of the shortest cycle with odd number of edges in $G$.

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**Introduction.** All graphs in this paper are finite, undirected, connected, have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. For a graph $G$, $\Delta(G)$ denotes the maximum degree of vertices in $G$. By $og(G)$ we denote the length of the shortest cycle with odd number of edges in $G$. The set of integers $\{a,a+1,\ldots,b\}$, $a \leq b$, is denoted by $[a,b]$. The terms, notations and concepts that we do not define can be found in [1].

A *proper edge-coloring* of graph $G$ is a coloring of the edges of $G$ such that no two adjacent edges receive the same color. The *chromatic index* $\chi'(G)$ of a graph $G$ is the minimum number of colors used in a proper edge-coloring of $G$. Vizing's theorem says that $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G)+1$ [2]. If $\alpha$ is a (partial) proper edge-coloring of $G$ and $v \in V(G)$, then the *spectrum* of a vertex $v$, denoted by $S(v,\alpha)$, is the set of colors of edges incident to $v$. By $\underline{S}(v,\alpha)$ and $\overline{S}(v,\alpha)$ we denote the smallest and largest colors of the spectrum respectively.

A proper edge-coloring of a graph $G$ with colors $1,2,\ldots,t$ is an *interval $t$-coloring*, if all colors are used and for any vertex $v$ of $G$ the set $S(v,\alpha)$ is an interval of integers [3,4]. A graph $G$ is *interval colorable*, if it has an interval $t$-coloring for some positive integer $t$. The set of interval colorable graphs is denoted by $\mathfrak{N}$. Trees, complete bipartite graphs, complete graphs with even number of vertices, $n$-dimensional cubes, various graph products are known to be interval colorable [5–9]. The following necessary condition was proved in [3,4,10].

***T h e o r e m A 1 .*** If $G$ is interval colorable, then $\chi'(G) = \Delta(G)$.

There exist many graphs that are not interval colorable. The simplest example is $K_3$. If $G$ is not interval colorable, then one can define a measure that determines how "far" the graph $G$ is from being interval colorable. The most studied such measure is called a *deficiency* of the graph, first defined in [11].

* E-mail: hrant.khachatrian@ysu.am, hrant@yerevann.com

***Definition 1.*** If $\alpha$ is a proper coloring of graph $G$, then deficiency of a vertex $v \in V(G)$ is defined as $def(v,\alpha) = \overline{S}(v,\alpha) - \underline{S}(v,\alpha) - |S(v,\alpha)| + 1$.

***Definition 2.*** Deficiency of a proper coloring $\alpha$ of graph $G$ is defined as

$$def(\alpha) = \sum_{v \in V(G)} def(v,\alpha).$$

***Definition 3.*** Deficiency of a graph $G$ is defined as $def(G) = \min\limits_{\alpha} def(\alpha)$, where minimum is taken over all proper colorings of $G$.

It easy to see that $def(G)$ is the minimum number of pendant edges that need to be attached to $G$ to make it interval colorable. Deficiency is studied for bipartite graphs,complete graphs, wheels, regular graphs, Hertz graphs and few other classes of graphs [11–14].

Another measure of "distance" of a graph from interval colorability is called *gap number*.

***Definition 4.*** If $\alpha$ is a proper coloring of $G$, then its gap number is defined as $gn(\alpha) = \max\limits_{v \in V(G)} def(v,\alpha)$.

***Definition 5.*** Gap number of a graph $G$ is defined as $gn(G) = \min\limits_{\alpha} gn(\alpha)$, where minimum is taken over all proper colorings of $G$.

The concept of gap number is directly related to interval $(t,h)$-colorings of graphs, that were introduced in [15] and studied in [16].

***Definition 6.*** $\alpha : E(G) \to \{1,\dots,t\}$ proper coloring is called an interval $(t,h)$-coloring of graph $G$, if all colors are used and for each vertex $v \in V(G)$ $def(v,\alpha) \le h$.

We denote by $\mathfrak{N}_t^h$ the set of all graphs that have interval $(t,h)$-colorings. $\mathfrak{N}^h = \bigcup_{t \ge 1} \mathfrak{N}_t^h$. It is easy to see that gap number of the graph $G$ is the minimum $h$, for which $G \in \mathfrak{N}^h$. If $G \in \mathfrak{N}^h$, then we say that $G$ is interval $h$-gap-colorable.

The following inequalities are immediately implied from the definitions:

$$gn(G) \le def(G) \le gn(G)|V(G)|.$$

In [11] it was shown that there exists a sequence of graphs $\{G_n\}$ such that $\lim\limits_{n\to\infty} def(G_n)/|V(G_n)| = 1$. On the other hand, no graph is known with deficiency more than the number of its vertices. One of the key open problems related to graph deficiency is the following conjecture:

***Conjecture.*** For any graph $G$, $def(G) \le |V(G)|$.

The conjecture is obviously true for interval 1-gap-colorable graphs. On the other hand, it is known that for every $h$ there exists a graph, which is not interval $h$-gap-colorable [17]. In this paper we prove the conjecture for outerplanar graphs.

**Related Work on Outerplanar Graphs.** Graph $G$ is called outerplanar if it can be drawn on the plane without crossings in such a way that all of its vertices belong to unbounded face of the drawing. Fiorini found the chromatic index of all outerplanar graphs [18].

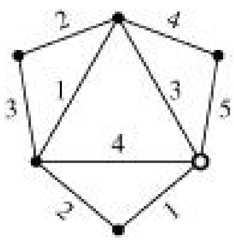


Fig. 1. An example of outerplanar triangulation that is not interval colorable. The coloring presented in the drawing has deficiency 1.

***Theorem A2.*** If $G$ is an outerplanar graph, then $\chi'(G) = \Delta(G) + 1$ if and only if $G$ is an odd cycle.

Theorem  immediately implies that odd cycles are not interval colorable. But there are many other outerplanar graphs that have positive deficiency. For example, the graph from Fig. 1 is Eulerian, has an odd number of edges and, according to Theorem 2 from [19], it cannot have an interval coloring. On the other hand, it is easy to see that this graph has a coloring with 1 deficiency.

Several subclasses of outerplanar graphs are known to have interval colorings. An outerplanar graph is called outerplanar triangulation, if every bounded face is a triangle. Triangular face of an outerplanar graph is called a separating triangle, if none of its edges belong to the unbounded face.

***Theorem A3.*** [20]. If $G$ is an outerplanar triangulation with more than three vertices and without separating triangles, then it is interval colorable.

The graph from Fig. 1. is an outerplanar triangluation containing a separating triangle, and it is not interval colorable.

***Theorem A4.*** [21]. If $G$ is a bipartite outerplanar graph, then it is interval colorable.

***Theorem A5.*** [22]. If $G$ is a subcubic outerplanar graph other than an odd cycle, then it is interval colorable.

**Main Results.** In this section we investigate the deficiency of all outerplanar graphs and generalize Theorem A2. Let $f_i(G)$ denote the number of faces having $i$ edges in the outerplanar graph $G$, $i = 3,4,\dots,|V(G)|$.

***Lemma.*** Let $G$ be a Hamiltonian outerplanar graph and let $w_0 \in V(G)$ be some fixed vertex. There exists a proper coloring $\alpha$ of $G$ such that:

- $def(w_0,\alpha) = 0$;
- $def(\alpha) = \sum\limits_{i\geq 3, i\,\text{odd}} f_i(G)$;
- $gn(\alpha) \leq f_3(G) + \operatorname{sgn} \sum\limits_{i\geq 5, i\,\text{odd}} f_i(G)$.

***Proof.*** We extend the method developed in [21, 23] to prove this Lemma. Let $|V(G)| = n$. We construct the coloring $\alpha$ by simultaneously coloring and labeling the edges of $G$. We denote the labeling function by $\lambda$: $\lambda : E(G) \to \{\boldsymbol{l},\boldsymbol{u},\boldsymbol{m}\}$. The algorithm visits one face of $G$ at each step and given the color and the label of one edge of the face uniquely determines the colors and the labels of the other edges. The order of visiting the faces is determined according to the so called weak dual graph $T$ of $G$. The vertices of $T$ correspond to the finite faces of $G$ and two vertices are joined with an edge, if the corresponding faces share an edge in $G$. It is known that the weak dual graph of an outerplanar graph is a forest [24]. In our case $G$ is Hamiltonian, hence 2-connected, so the weak dual graph is a tree. The algorithm for constructing $\alpha$ and $\lambda$ is described below.

1. Draw the graph $G$ on a plane without crossings in a way that all its vertices are on the unbounded face. The edges of the unbounded face form a Hamiltonian cycle. Denote the vertices along that cycle starting from the $w_0$ vertex by $w_0,w_1,\dots,w_{n-1}$ in the clockwise direction.

2. Compute the weak dual tree $T$ of $G$.

3. Take the face of $G$ that contains the edge $w_0w_{n-1}$. Let $t_0 \in V(T)$ be the vertex corresponding to that face. Note that this face is uniquely determined (otherwise, the edge $w_0w_{n-1}$ would not be on the outer face). Let the vertices of that face be $w_0 = v_1,v_2,\dots,v_r = w_{n-1}$ (in the clockwise direction).

4. Set $\alpha(v_1v_r) = 1$ and $\lambda(v_1v_r) = \boldsymbol{u}$.

5. Traverse the tree $T$ using breadth-first search, starting from the vertex $t_0$. Let $F$ be the face of $G$ corresponding to the current vertex $t \in V(T)$. The algorithm guarantees that one of the edges of $F$ is already colored and labeled. Let $V(F) = \{v_1, v_2, \dots, v_r\}$ (in the clockwise direction), where $v_1 v_r$ is the labeled edge and $\alpha(v_1 v_r) = k$. The colors and labels of the remaining edges are determined depending on $r$ and the label $\lambda(v_1 v_r)$ (Fig. 2):

   (a) If $\lambda(v_1 v_r) = \boldsymbol{l}$, then the algorithm guarantees that $\underline{S}(v_1, \alpha) = \underline{S}(v_r, \alpha) = k$, so if we use smaller colors on the new edges, the coloring will remain proper.

      i. If $r = 3$, then set $\alpha(v_1 v_2) = k-1$, $\alpha(v_2 v_3) = k-2$, $\lambda(v_1 v_2) = \boldsymbol{m}$ and $\lambda(v_2 v_3) = \boldsymbol{l}$. Note that the color $k-1$ will be missing in the spectrum of $v_3$ and it will not be filled in the later steps of the algorithm.

      ii. If $r = 2s$, $s \geq 2$, then color the edges $v_1 v_2, v_2 v_3, \dots, v_{2s-1} v_{2s}$ by alternating $k-1$ and $k$, and label them by alternating $\boldsymbol{l}$ and $\boldsymbol{u}$.

      iii. If $r = 2s+1$, $s \geq 2$, then set $\alpha(v_1 v_2) = k-1$, $\alpha(v_2 v_3) = k-2$, $\lambda(v_1 v_2) = \boldsymbol{m}$, $\lambda(v_2 v_3) = \boldsymbol{l}$, color the edges $v_3 v_4, v_4 v_5, \dots, v_{2s} v_{2s+1}$ by alternating $k$ and $k-1$, and label them by alternating $\boldsymbol{u}$ and $\boldsymbol{l}$. Note that the color $k-1$ is missing from the spectrum of $v_3$.

   (b) If $\lambda(v_1 v_r) = \boldsymbol{u}$, then the algorithm guarantees that $\overline{S}(v_1, \alpha) = \overline{S}(v_r, \alpha) = k$, so if we use larger colors on the new edges, the coloring will remain proper.

      i. If $r = 3$, then set $\alpha(v_1 v_2) = k+1$, $\alpha(v_2 v_3) = k+2$, $\lambda(v_1 v_2) = \boldsymbol{m}$ and $\lambda(v_2 v_3) = \boldsymbol{u}$. Note that the color $k+1$ is missing in the spectrum of $v_3$.

      ii. If $r = 2s$, $s \geq 2$, then color the edges $v_1 v_2, v_2 v_3, \dots, v_{2s-1} v_{2s}$ by alternating $k-1$ and $k$, and label them by alternating $\boldsymbol{u}$ and $\boldsymbol{l}$.

      iii. If $r = 2s+1$, $s \geq 2$, then set $\alpha(v_1 v_2) = k+1$, $\alpha(v_2 v_3) = k+2$, $\lambda(v_1 v_2) = \boldsymbol{m}$, $\lambda(v_2 v_3) = \boldsymbol{u}$, color the edges $v_3 v_4, v_4 v_5, \dots, v_{2s} v_{2s+1}$ by alternating $k$ and $k+1$, and label them by alternating $\boldsymbol{l}$ and $\boldsymbol{u}$. Note that the color $k+1$ is missing from the spectrum of $v_3$.

   (c) If $\lambda(v_1 v_r) = \boldsymbol{m}$, then the algorithm guarantees that either $\overline{S}(v_1, \alpha) = \underline{S}(v_r, \alpha) = k$ or $\underline{S}(v_1, \alpha) = \overline{S}(v_r, \alpha) = k$.

      i. If $\underline{S}(v_1, \alpha) = \overline{S}(v_r, \alpha) = k$, then rename the vertices of $F$ in the counter-clockwise direction, so we guarantee that $\overline{S}(v_1, \alpha) = \underline{S}(v_r, \alpha) = k$.

      ii. If $r = 3$, then set $\alpha(v_1 v_2) = k+1$, $\alpha(v_2 v_3) = k-1$, $\lambda(v_1 v_2) = \boldsymbol{u}$ and $\lambda(v_2 v_3) = \boldsymbol{l}$. Note that the color $k$ will be missing in the spectrum of $v_2$.

      iii. If $r = 2s$, $s \geq 2$, then set $\alpha(v_1 v_2) = k+1$, $\alpha(v_2 v_3) = k$, $\lambda(v_1 v_2) = \boldsymbol{u}$ and $\lambda(v_2 v_3) = \boldsymbol{m}$. Then color the edges $v_3 v_4, v_4 v_5, \dots, v_{2s-1} v_{2s}$ by alternating $k-1$ and $k$, and label them by alternating $\boldsymbol{u}$ and $\boldsymbol{l}$.

      iv. If $r = 2s+1$, $s \geq 2$, then set $\alpha(v_1 v_2) = k+1$, $\lambda(v_1 v_2) = \boldsymbol{u}$, color the edges $v_2 v_3, v_3 v_4, \dots, v_{2s} v_{2s+1}$ by alternating $k-1$ and $k$, and label them by alternating $\boldsymbol{l}$ and $\boldsymbol{u}$.

6. If $\min\limits_{e \in E(G)} \alpha(e) = c_0 \neq 1$, then adjust the colors on all edges by the formula $\alpha(e) = \alpha(e) - c_0 + 1$ for all $e \in E(G)$.

To complete the proof we have to check the three statements of the Lemma. First, note that the algorithm introduces exactly one missing color in the spectrum of one of the vertices of every face with odd number of edges (steps 5(a) i, 5(b) i, 5(c) ii, 5(a) iii, 5(b) iii and 5(c) iv).

Moreover, the algorithm doesn't introduce any missing colors in the vertices of the faces with even number of edges (steps 5(a) ii, 5(b) ii, 5(c) iii). This implies that the total number of missing colors in the spectrums equals the number of faces with odd number of edges: $def(\alpha) = \sum\limits_{i\geq 3, i \text{ odd}} f_i(G)$.

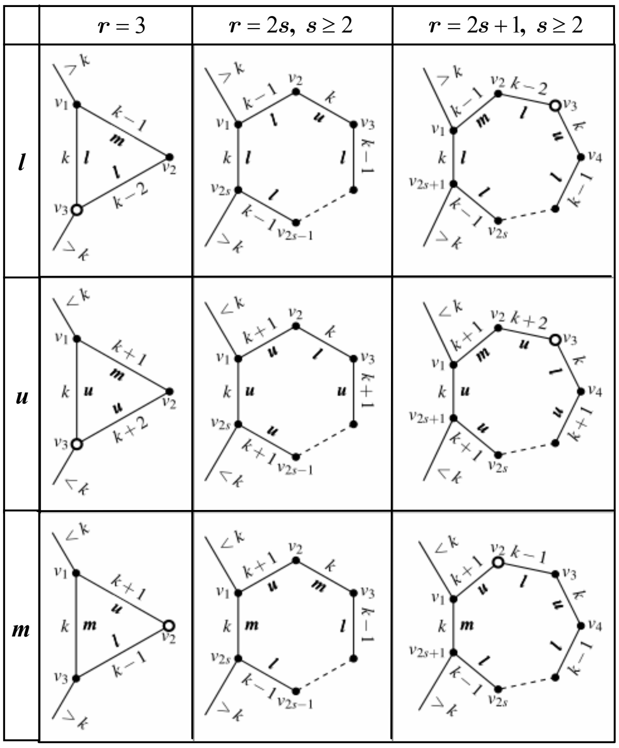


Fig. 2. Description of the step 5 of the algorithm described in the proof of Lemma.

Rows correspond to the label of the precolored edge $v_1v_r$.

Columns correspond to the number of edges of the current face $F$.

Vertices denoted by circles (instead of disks) indicate that one more color will miss from their spectrum after the current iteration of the algorithm.

Next, it is important to note that the deficiency of any vertex can be more than one only in the case when the algorithm introduces a missing color several times while processing different faces containing that vertex. When $r \geq 5$ (steps 5(a) iii, 5(b) iii and 5(c) iv) or when $r = 3$ and $\lambda(v_1v_3) = \boldsymbol{m}$ (step 5(d) ii) the missing color is introduced to a vertex that was never visited before. So the deficiency can become larger than one only during the steps 5(a) i and 5(b) i ($r = 3$ and $\lambda(v_1v_r) = \boldsymbol{l}$ or $\lambda(v_1v_r) = \boldsymbol{u}$). Thus the deficiency of any vertex cannot be more than the number of triangles in $G$ plus one, in that case it also belongs to a larger odd cycle. Therefore, $gn(\alpha) = \max\limits_{v\in V(G)} \{def(v,\alpha)\} \leq f_3(G) + \operatorname{sgn} \sum\limits_{i\geq 5, i \text{ odd}} f_i(G)$. Equality is achieved in the case the graph $G$ is a fan.

Finally, we need to show that $def(w_0,\alpha) = 0$. Suppose it belongs to $k \geq 1$ different faces. These faces are visited in the order defined by the traversal of the tree $T$ (which is a breadth-first search). After the step 4 each time one of these $k$ faces is visited. The vertex $w_0$ will belong to the edge that is already colored and labeled. It will be denoted by $v_1$. The only exception is in the step 5(c)i, when $\lambda(v_1v_r) = \boldsymbol{m}$ and $w_0$ might be denoted by $v_r$. Note that regardless of the length of the face $r$ and the initial label $\lambda(v_1v_r)$, no missing colors are introduced at the vertex $v_1$. $\square$

To prove the next theorem we need the concept of block-cut-vertex tree. Block is defined as a maximal 2-connected subgraph. Cut vertex is a vertex whose removal increases the number of connected components of the graph. For a given graph $G$ let $B$ denote the set of blocks and $C$ denote the set of cut vertices. Note that every cut vertex belongs to at least two blocks. We can construct a graph $bc(G)$ in the following way: $V(bc(G)) = B \cup C$ and the vertices $b \in B$ and $c \in C$ are joined with an edge if and only if the vertex $c$ belongs to the block $b$. It is easy to see that $bc(G)$ is a tree [25] and is called a block-cut-vertex tree of $G$.

***Theorem 1.*** If $G$ is an outerplanar graph, then

$$def(G) \le \sum_{i\ge 3, i\,\text{odd}} f_i(G) \quad\text{and}\quad gn(G) \le f_3(G) + \operatorname{sgn} \sum_{i\ge 5, i\,\text{odd}} f_i(G).$$

***Proof.*** Let $bc(G)$ denote the block-cut-vertex tree of $G$. Denote the blocks of $G$ by $B_1, B_2, \ldots, B_m$, $m \ge 1$, and cut vertices by $c_1, \ldots c_n$, $n \ge 0$. The blocks are either isomorphic to $K_2$ or are Hamiltonian outerplanar graphs. We construct a coloring $\beta$ of the graph $G$ based on the colorings of blocks. We start from the vertex $B_1$ and color the corresponding block. If $B_1$ is isomorphic to $K_2$, we color its only edge by 1. If it is a Hamiltonian outerplanar graph, we set $\beta(e) = \alpha_1(e)$ for every $e \in E(B_1)$, where $\alpha_1$ is the coloring of $B_1$ from Lemma.

Next, we traverse the tree $bc(G)$ using breadth-first search. Suppose the block $B_i \in V(bc(G))$ is reached. Its parent in $bc(G)$ is some cut vertex $c_k$ whose parent is another, already colored, block $B_j$. Suppose that at this stage we have $\overline{S}(c_k, \beta) = t$. We construct a coloring $\alpha_i$ of the block $B_i$. If $B_i$ is isomorphic to $K_2$, we color its only edge by 1. Otherwise, $\alpha_i$ is the coloring of $B_i$ from Lemma by setting $w_0 = c_k$, so that $def(c_k, \alpha_i) = 0$. We color the corresponding edges in $G$ by setting $\beta(e) = \alpha_i(e) + t$ for all $e \in E(B_i)$. So the equality of $def(c_k, \beta)$ and $def(c_k, \alpha_j)$ after coloring $B_i$ is guaranteed.

It is easy to see that $def(\beta) = \sum_k def(\alpha_k) = \sum_k \sum_{\substack{i\ge 3\\ i\,\text{odd}}} f_i(B_k) = \sum_{\substack{i\ge 3\\ i\,\text{odd}}} f_i(G),$

$$gn(\beta) = \max_k gn(\alpha_k) \le \max_k \left\{ f_3(B_k) + \operatorname{sgn} \sum_{\substack{i\ge 5\\ i\,\text{odd}}} f_i(B_k) \right\} \le f_3(G) + \operatorname{sgn} \sum_{\substack{i\ge 5\\ i\,\text{odd}}} f_i(G). \ \square$$

***Corollary 1.*** If $G$ is a bipartite outerplanar graph, then it is interval colorable.

***Corollary 2.*** If $G$ is a triangle free outerplanar graph, then it is interval 1-gap-colorable.

***Corollary 3.*** If $G$ is an outerplanar graph, then

$$def(G) \le (|V(G)| - 2)/(og(G) - 2).$$

***Proof.*** Every Hamiltonian outerplanar graph can be constructed by starting from $K_2$ and iteratively adding bounded faces. Every new bounded face with $m$ edges adds exactly $m-2$ vertices. So, if $G$ is Hamiltonian, we get $|V(G)| = 2 + \sum_{i\ge 3} f_i(G)(i-2)$. In the general case, when $B_1, \ldots, B_k$, $k \ge 1$, are the Hamiltonian blocks of $G$, we have

$$|V(G)| \ge 1 + \sum_{j=1}^{k} (|V(B_j)| - 1) \ge 1 + \sum_{j=1}^{k} \left( 2 + \sum_{i\ge 3} f_i(B_j)(i-2) - 1 \right)$$
$$\ge 2 + \sum_{i\ge 3} f_i(G)(i-2) \ge 2 + \sum_{i\ge 3, i\,\text{odd}} f_i(G)(i-2) \ge 2 + \sum_{i\ge 3, i\,\text{odd}} f_i(G)(og(G)-2).$$

The last inequality holds, because of the definition of $og(G)$. Due to Theorem 6, we obtain: $def(G) \le \sum_{i\ge 3, i\ \text{odd}} f_i(G) \le \dfrac{|V(G)| - 2}{og(G) - 2}$. $\square$

This means that the outerplanar graphs satisfy Conjecture.

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