M a t h e m a t i c s

EMBEDDING THEOREMS FOR MULTIANISOTROPIC SPACES WITH TWO VERTICES OF ANISOTROPICITY

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We prove embedding theorems for multianisotropic spaces in the case when the Newton polyhedron has two vertices of anisotropicity. The case of one anisotropicity vertex of the polyhedron was studied in previous papers of one of the authors. The present paper is the continuation of those.

Introduction. The history of embedding theorems begins with the works of S.L. Sobolev (see [1, 2]) and later these results for anisotropic spaces were continued by different mathematicians. In the papers [3–9] one can find the history of the problem and different results. This paper is a continuation of [10–12], where we proved embedding theorems for multianisotropyc spaces for the characteristic polyhedron having one vertex of anisotropicity. Below we study same problem for the polyhedron with two anisotropicity vertices. The difficulty of the study of two vertices case is the selection of a "dominant" facet, which is done in the work.

Multianisotropic Exponents and Their Properties. Let $\mathbb{R}^n$ be n-dimensional space, $\mathbb{Z}_+^n$ the set of all multi-indices. For $\xi, \eta \in \mathbb{R}^n$, $\alpha \in \mathbb{Z}_+^n$, $t > 0$ let $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$, $t^\eta = (t^{\eta_1}, \ldots, t^{\eta_n})$, $D_k = \frac{1}{i} \cdot \frac{\partial}{\partial x_k}$ $(k = 1, \ldots, n)$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ be the weak derivative. Let $\mathfrak{N}$ be the completely regular polyhedron (f.e., see [10]) with vertices $l^0 = (0, 0, \ldots, 0)$, $l^1 = (l_1, 0, \ldots, 0)$, $l^2 = (0, l_2, 0, \ldots, 0)$, $\ldots$, $l^n = (0, 0, \ldots, 0, l_n)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ such that α and β have only positive coordinates. Below we denote these vertices by $\{\alpha^1, \alpha^2, \ldots, \alpha^{n+2}\}$. Let μ^i $(i = 1, \ldots, 2n-1)$ be the outer normal of the $(n-1)$-dimensional non-coordinate face $\mathfrak{N}_i^{n-1}$ $(i = 1, \ldots, 2n-1)$ such that the equation of the $(n-1)$-dimensional hyperplane, containing this face is $(\alpha, \mu^i) = 1$ $(i = 1, \ldots, 2n-1)$. For a parameter $\nu > 0$ and a natural number k denote

$$P(\nu, \xi) = \sum_{i=1}^{n+2} \left(\nu \xi^{\alpha^i} \right)^{2k}, \tag{1}$$

$$G_0(\nu, \xi) = e^{-P(\nu, \xi)}, \tag{2}$$

$$G_{1,j}(\nu, \xi) = 2k \left(\nu \xi^{\alpha^j} \right)^{2k-1} e^{-P(\nu, \xi)}, \quad j = 1, \ldots, n+2, \tag{3}$$

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and let $\hat{G}_0(t,v)$, $\hat{G}_{1,j}(t,v)$ $(j = 1,\ldots,n+2)$ be the corresponding Fourier transforms of these functions. It is obvious that $G_0, G_{1,j}, \hat{G}_0, \hat{G}_{1,j} \in S$, where $S = S(\mathbb{R}^n)$ is the Schwartz space of functions infinitely differentiable and rapidly decreasing at infinity.

For any multi-index $m = (m_1, \ldots, m_n)$ we denote

$$I = I(v) = \int_{\mathbb{R}^n} \xi_1^{m_1} \ldots \xi_n^{m_n} e^{-P(v,\xi)} d\xi_1 \ldots d\xi_n \tag{4}$$

and study the behavior of the function $I(v)$ depending on $v : 0 < v < 1$.

L e m m a 1. For any multi-index $m = (m_1, \ldots, m_n)$ there exist constants $C_0, \ldots, C_n$, such that for any $v : 0 < v < 1$ the following inequality holds:

$$|I(v)| \leq (C_n|\ln v|^n + \cdots + C_1|\ln v| + C_0) v^{-\max\limits_{1 \leq i \leq 2n-1} |\mu^i|}. \tag{5}$$

P r o o f. First we prove Lemma 1 for $n = 3$.

Consider the ratios $\dfrac{\alpha_i}{m_i + 1}$ $(i = 1, 2, 3)$ and denote by i_0 one of the indices, for which the ratio is maximal, i.e. $\max\limits_{i=1,2,3} \dfrac{\alpha_i}{m_i + 1} = \dfrac{\alpha_{i_0}}{m_{i_0} + 1}$. Let $i_0 = 3$ and all other rations are less than $\dfrac{\alpha_3}{m_3 + 1}$. Consider the facet passing through the vertices $(l_1, 0, 0)$, $(0, l_2, 0)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Let this face has the outer normal μ^4. Substituting in Eq. (5) $\xi = v^{-\mu^4} \eta$, we obtain

$$|I(v)| \leq C v^{-\left(|\mu^4| + (m,\mu^4)\right)} \int_0^\infty e^{-\left(\eta_1^{\frac{\alpha_1}{\alpha_3}} \eta_2^{\frac{\alpha_2}{\alpha_3}} \eta_3\right)^{2k\alpha_3}} \left(\eta_1^{\frac{\alpha_1}{\alpha_3}} \eta_2^{\frac{\alpha_2}{\alpha_3}} \eta_3\right)^{m_3} d\left(\eta_1^{\frac{\alpha_1}{\alpha_3}} \eta_2^{\frac{\alpha_2}{\alpha_3}} \eta_3\right) \times$$

$$\times \int_0^\infty \eta_1^{m_1 - \frac{\alpha_1}{\alpha_3} m_3 - \frac{\alpha_1}{\alpha_3}} e^{-\eta_1^{2kl_1}} d\eta_1 \int_0^\infty \eta_2^{m_2 - \frac{\alpha_2}{\alpha_3} m_3 - \frac{\alpha_2}{\alpha_3}} e^{-\eta_2^{2kl_2}} d\eta_2 \leq C v^{-\max\limits_{i=1,\ldots,5}\left(|\mu^i| + (m,\mu^i)\right)}.$$

The last relation follows from the convergence of three integrals and from the inequalities $m_1 - \dfrac{\alpha_1}{\alpha_3} m_3 - \dfrac{\alpha_1}{\alpha_3} > -1$, $m_2 - \dfrac{\alpha_2}{\alpha_3} m_3 - \dfrac{\alpha_2}{\alpha_3} > -1$. The cases $\dfrac{\alpha_1}{m_1 + 1} < \dfrac{\alpha_2}{m_2 + 1} = \dfrac{\alpha_3}{m_3 + 1}$ and $\dfrac{\alpha_1}{m_1 + 1} = \dfrac{\alpha_2}{m_2 + 1} = \dfrac{\alpha_3}{m_3 + 1}$ are studied, in the same way as in [3] were done.

Suppose $i_0 \neq 3$ and consider similar rations for the other vertex of anisotropicity.

Let j_0 be one of the indices, for which the ratio is maximal, i.e. $\max\limits_{j=1,2,3} \dfrac{\beta_j}{m_j + 1} = \dfrac{\beta_{j_0}}{m_{j_0} + 1}$. Now for $j_0 = 2$ consider the facet passing through the vertices $(l_1, 0, 0)$, $\beta = (\beta_1, \beta_2, \beta_3)$, $(0, 0, l_3)$ with the outer normal μ^2. For $j_0 = 1$ we consider the facet with the vertices $\beta = (\beta_1, \beta_2, \beta_3)$, $(0, l_2, 0)$, $(0, 0, l_3)$ and with the outer normal μ^1. Then the substitutions $\xi = v^{-\mu^2} \eta$ or $\xi = v^{-\mu^1} \eta$ in (5) respectively yeild the same result with a slight change of the form of the integral (for example, l_3 instead of l_2 or α instead of β and etc.).

The only cases we need to consider are:

a) $\max\limits_{i=1,2,3} \dfrac{\alpha_i}{m_i + 1} = \dfrac{\alpha_2}{m_2 + 1}$ and $\max\limits_{j=1,2,3} \dfrac{\beta_j}{m_j + 1} = \dfrac{\beta_3}{m_3 + 1}$;

b) $\max\limits_{i=1,2,3} \dfrac{\alpha_i}{m_i + 1} = \dfrac{\alpha_1}{m_1 + 1}$ and $\max\limits_{j=1,2,3} \dfrac{\beta_j}{m_j + 1} = \dfrac{\beta_3}{m_3 + 1}$.

We only study a), since it is symmetric to b).

At first we consider the case, when the maximum is unique (the equality in these relations is discussed later), i.e.

$$\frac{\alpha_1}{\alpha_2} < \frac{m_1 + 1}{m_2 + 1}, \frac{\alpha_3}{\alpha_2} < \frac{m_3 + 1}{m_2 + 1}, \quad \frac{\beta_1}{\beta_3} < \frac{m_1 + 1}{m_3 + 1}, \frac{\beta_2}{\beta_3} < \frac{m_2 + 1}{m_3 + 1}. \tag{6}$$

Substituting $\xi = v^{-\mu^4}\eta$ in Eq. (5), we obtain

$$|I(v)| \le C v^{-\left(|\mu^4|+(m,\mu^4)\right)} \iiint_{\mathbb{R}^3_+} \eta_1^{m_1-\frac{\alpha_1}{\alpha_2}-\frac{\beta_1}{\beta_3}} \eta_2^{m_2-\frac{\beta_2}{\beta_3}} \eta_3^{m_3-\frac{\alpha_3}{\alpha_2}} \times$$

$$\times e^{-\eta_1^{2kl_1}-\left(\eta_1^{\frac{\alpha_1}{\alpha_2}}\eta_2\eta_3^{\frac{\alpha_3}{\alpha_2}}\right)^{2k\alpha_2}-\left(\eta_1^{\frac{\beta_1}{\beta_3}}\eta_2^{\frac{\beta_2}{\beta_3}}\eta_3\right)^{2k\beta_3}} d\eta_1 \, d\left(\eta_1^{\frac{\alpha_1}{\alpha_2}}\eta_2\eta_3^{\frac{\alpha_3}{\alpha_2}}\right) d\left(\eta_1^{\frac{\beta_1}{\beta_3}}\eta_2^{\frac{\beta_2}{\beta_3}}\eta_3\right).$$

Let K, M, L be numbers satisfying the relation

$$\eta_1^K \left(\eta_1^{\frac{\alpha_1}{\alpha_2}}\eta_2\eta_3^{\frac{\alpha_3}{\alpha_2}}\right)^M \left(\eta_1^{\frac{\beta_1}{\beta_3}}\eta_2^{\frac{\beta_2}{\beta_3}}\eta_3\right)^L = \eta_1^{m_1-\frac{\alpha_1}{\alpha_2}-\frac{\beta_1}{\beta_3}} \eta_2^{m_2-\frac{\beta_2}{\beta_3}} \eta_3^{m_3-\frac{\alpha_3}{\alpha_2}}.$$

By equating the corresponding exponents of η_i $(i = 1, 2, 3)$, we get a system of linear equations with respect to the unknowns K, M, L. If we denote $x = K+1$, $y = M+1$ and $z = L+1$, then the equations take the following form:

$$(7) \qquad \begin{cases} x + \dfrac{\alpha_1}{\alpha_2}y + \dfrac{\beta_1}{\beta_3}z = m_1 + 1, \\[2mm] y + \dfrac{\beta_2}{\beta_3}z = m_2 + 1, \\[2mm] \dfrac{\alpha_3}{\alpha_2}y + z = m_3 + 1. \end{cases}$$

By substituting $\xi = v^{-\mu^3}\eta$ and applying similar steps, we get the following system of linear equations:

$$(8) \qquad \begin{cases} x + \dfrac{\beta_1}{\beta_3}z = m_1 + 1, \\[2mm] \dfrac{\alpha_2}{\alpha_1}x + y + \dfrac{\beta_2}{\beta_3}z = m_2 + 1, \\[2mm] \dfrac{\alpha_3}{\alpha_1}x + z = m_3 + 1. \end{cases}$$

So we need to show that either one of Eqs. (7) and (8) has a non-negative solution in order to ensure that $K, M, L \ge -1$.

Using the conditions (6), it is easy to check that y and z are positive in Eq. (7).

Consider x. By Cramer's rule and the corresponding substitution, we have

$$x = \dfrac{\begin{vmatrix} m_1+1 & \alpha_1/\alpha_2 & \beta_1/\beta_3 \\ m_2+1 & 1 & \beta_2/\beta_3 \\ m_3+1 & \alpha_3/\alpha_2 & 1 \end{vmatrix}}{1 - (\alpha_3/\alpha_2) \cdot (\beta_2/\beta_3)} = \dfrac{\begin{vmatrix} m_1+1 & m_2+1 & m_3+1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}}{\alpha_2\beta_3 - \alpha_3\beta_2} = \dfrac{\Delta}{\alpha_2\beta_3 - \alpha_3\beta_2}.$$

Note that, due to conditions (6), we have $\alpha_2\beta_3 - \alpha_3\beta_2 > 0$, so x and Δ have the same sign. Let Δ_1, Δ_2 and Δ_3 be the corresponding cofactors of Laplace expansion of the determinant Δ along the first row, i.e. $\Delta_1 = \alpha_2\beta_3 - \alpha_3\beta_2$, $\Delta_2 = \alpha_3\beta_1 - \alpha_1\beta_3$ and $\Delta_3 = \alpha_1\beta_2 - \alpha_2\beta_1$. Thus, $\Delta = (m_1+1)\Delta_1 + (m_2+1)\Delta_2 + (m_3+1)\Delta_3$. Since the conditions (6) hold, we have $\Delta_1 > 0$. Notice that $\alpha_1\Delta_1 + \alpha_2\Delta_2 + \alpha_3\Delta_3 = \beta_1\Delta_1 + \beta_2\Delta_2 + \beta_3\Delta_3 = 0$ and each Δ_i cannot be positive. Consider the following cases:

1. $\Delta_3 \ge 0$.

By (6) we have $m_1 + 1 > \dfrac{m_2+1}{\alpha_2}\alpha_1$ and $m_3 + 1 > \dfrac{m_2+1}{\alpha_2}\alpha_3$. Applying these inequalities to the expansion, we obtain

$$\Delta = (m_1+1)\Delta_1 + (m_2+1)\Delta_2 + (m_3+1)\Delta_3 > \dfrac{m_2+1}{\alpha_2}(\Delta_1\alpha_1 + \Delta_2\alpha_2 + \Delta_3\alpha_3) = 0.$$

2. $\Delta_3 < 0$ and $\Delta_2 \geq 0$.

As in the previous case, from Eq. (6) we obtain $\Delta > 0$.

3. $\Delta_3 < 0$ and $\Delta_2 < 0$.

Consider $\dfrac{\alpha_1}{\alpha_3}$ and $\dfrac{\beta_1}{\beta_2}$. By (4) we also have $m_2 + 1 < \dfrac{m_1 + 1}{\alpha_1} \alpha_2$, $m_3 + 1 < \dfrac{m_1 + 1}{\beta_1} \beta_3$.

Suppose that either $\dfrac{\alpha_1}{\alpha_3} \leq \dfrac{m_1 + 1}{m_3 + 1}$ or $\dfrac{\beta_1}{\beta_2} \leq \dfrac{m_1 + 1}{m_2 + 1}$ holds. Then either $m_3 + 1 \leq \dfrac{m_1 + 1}{\alpha_1} \alpha_3$ or $m_2 + 1 \leq \dfrac{m_1 + 1}{\beta_1} \beta_2$. The both cases can be proceed analogously to the cases 1 and 2.

It remains to consider the case when $\dfrac{\alpha_3}{\alpha_1} < \dfrac{m_3 + 1}{m_1 + 1}$, $\dfrac{\beta_2}{\beta_1} < \dfrac{m_2 + 1}{m_1 + 1}$, $\Delta_2 < 0$ and $\Delta_3 < 0$, where in the case $\Delta < 0$ consider (8), where x and z are positive. For y we have:

$$
y = \frac{\begin{vmatrix} 1 & m_1 + 1 & \beta_1/\beta_3 \\ \alpha_2/\alpha_1 & m_2 + 1 & \beta_2/\beta_3 \\ \alpha_3/\alpha_1 & m_3 + 1 & 1 \end{vmatrix}}{1 - (\alpha_3/\alpha_1) \cdot (\beta_1/\beta_3)} = \frac{\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ m_1 + 1 & m_2 + 1 & m_3 + 1 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}}{\alpha_2 \beta_1 - \alpha_1 \beta_2} = \frac{\Delta}{\Delta_2}.
$$

Hence, in this case the signs since of y in Eq. (8) and x in Eq. (7) are of different. Thus, if $\Delta > 0$, then the solution of (7) is positive. If $\Delta < 0$, then the solution of (8) is positive. If $\Delta = 0$, then the solutions of each equations are non-negative. We split the integral into four parts and estimate each one separately. Let $\mu_i^0 = \min\limits_{1 \leq j \leq 5} \mu_i^j$ $(i = 1, 2)$, then we have

$$
|I(v)| \leq \int_0^{v^{-\mu_1^0}} d\xi_1 \int_0^{v^{-\mu_2^0}} d\xi_2 \int_0^\infty \ldots d\xi_3 + \int_{v^{-\mu_1^0}}^0 d\xi_1 \int_0^{v^{-\mu_2^0}} d\xi_2 \int_0^\infty \ldots d\xi_3
$$

$$
+ \int_0^{v^{-\mu_1^0}} d\xi_1 \int_{v^{-\mu_2^0}}^0 d\xi_2 \int_0^\infty \ldots d\xi_3 + \int_0^{v^{-\mu_1^0}} d\xi_1 \int_0^{v^{-\mu_2^0}} d\xi_2 \int_0^\infty \ldots d\xi_3
$$

$$
= I_1 + I_2 + I_3 + I_4.
$$

By substituting $\xi = v^{-\mu^1} \eta$ in I_1, we get

$$
I_1 \leq C v^{-\left(|\mu^1| + (m, \mu^1)\right)} \int_0^1 d\eta_1 \int_0^1 d\eta_2 \int_0^\infty \eta^m e^{-\eta_2^{2kl_2} - \eta_3^{2kl_3} - \eta^{2k\beta}} d\eta_3
$$

$$
\leq C v^{-\left(|\mu^1| + (m, \mu^1)\right)} \int_0^\infty \eta_3^{m_3} e^{-\eta_3^{2kl_3}} d\eta_3 \leq C v^{-\left(|\mu^1| + (m, \mu^1)\right)}.
$$

Since the last integral is non-negative, then $0 \leq \eta_1 \leq v^{\mu_1^1 - \mu_1^0} \leq 1$ and $0 \leq \eta_2 \leq v^{\mu_2^1 - \mu_2^0} \leq 1$ if $0 \leq \xi_1 \leq v^{-\mu_1^0}$ and $0 \leq \xi_2 \leq v^{-\mu_2^0}$.

Similarly, one can estimate I_2, I_3 and I_4, considering the solutions of (8) and (7) with the corresponding substitutions $\xi = v^{-\mu^2} \eta$, $\xi = v^{-\mu^3} \eta$ and $\xi = v^{-\mu^2} \eta$.

As a result in this case we obtain

$$
|I(v)| \leq (C_1 \ln v + C_2) v^{-\max\limits_{i=1,\ldots,5}\left(|\mu^i| + (m, \mu^i)\right)}.
$$

Now we return to the case of equality in Eq. (6) there is an equal sign. If $\dfrac{\alpha_1}{\alpha_2} = \dfrac{m_1 + 1}{m_2 + 1}$ and $\dfrac{\alpha_3}{\alpha_2} < \dfrac{m_3 + 1}{m_2 + 1}$, then the solution of (7) is non-negative and the integral can be estimated as in the case $\Delta = 0$. If $\dfrac{\alpha_1}{\alpha_2} < \dfrac{m_1 + 1}{m_2 + 1}$ and $\dfrac{\alpha_3}{\alpha_2} = \dfrac{m_3 + 1}{m_2 + 1}$, then $\dfrac{\alpha_1}{\alpha_3} < \dfrac{m_1 + 1}{m_3 + 1}$. This case has already been discussed. If $\dfrac{\beta_1}{\beta_3} = \dfrac{m_1 + 1}{m_3 + 1}$ and $\dfrac{\beta_2}{\beta_3} < \dfrac{m_2 + 1}{m_3 + 1}$, then $\dfrac{\beta_2}{\beta_1} < \dfrac{m_2 + 1}{m_1 + 1}$. This case is

handled in the same way as $j_0 = 1$. If $\dfrac{\beta_1}{\beta_3} < \dfrac{m_1 + 1}{m_3 + 1}$ and $\dfrac{\beta_2}{\beta_3} < \dfrac{m_2 + 1}{m_3 + 1}$, then $\dfrac{\beta_1}{\beta_2} < \dfrac{m_1 + 1}{m_2 + 1}$, so we can apply the case $j_0 = 2$.

Now we prove the Lemma 1 for any n.

We cut the completely regular polyhedron $\mathfrak{N}$ with the hyperplane $(l^1, l^2, \ldots, l^{n-1}, \alpha)$. Now we consider the polyhedron $\mathfrak{M}$ with vertices $(l^1, \ldots, l^n, \alpha)$, which has one anisotropicity vertex α. Now, if for some i $(i = 1, \ldots, n)$ we apply the substitution $\xi = v^{-\mu^i}\eta$ to (5), we obtain

$$|I(v)| \leq Cv^{-\left(|\mu^i| + (m, \mu^i)\right)} \times$$

$$\times \int_{\mathbb{R}^n} e^{-\eta_1^{2kl_1} - \cdots - \left(\eta_1^{\frac{\alpha_1}{\alpha_i}} \cdots \eta_{i-1}^{\frac{\alpha_{i-1}}{\alpha_i}} \eta_i \eta_{i+1}^{\frac{\alpha_{i+1}}{\alpha_i}} \cdots \eta_n^{\frac{\alpha_n}{\alpha_i}}\right)^{2k\alpha_i} - \cdots - \eta_n^{2kl_n}} \eta_1^{m_1 - \frac{\alpha_1}{\alpha_i}} \cdots \eta_{i-1}^{m_{i-1} - \frac{\alpha_{i-1}}{\alpha_i}} \eta_i^{m_i} \times$$

$$\times \eta_{i+1}^{m_{i+1} - \frac{\alpha_{i+1}}{\alpha_i}} \cdots \eta_n^{m_n - \frac{\alpha_n}{\alpha_i}} d\eta_1 \ldots d\left(\eta_1^{\frac{\alpha_1}{\alpha_i}} \cdots \eta_{i-1}^{\frac{\alpha_{i-1}}{\alpha_i}} \eta_i \eta_{i+1}^{\frac{\alpha_{i+1}}{\alpha_i}} \cdots \eta_n^{\frac{\alpha_n}{\alpha_i}}\right) \ldots d\eta_n.$$

Let $k_1, k_2, \ldots, k_n$ be numbers satisfying the following relation:

$$\eta_1^{k_1} \cdots \left(\eta_1^{\frac{\alpha_1}{\alpha_i}} \cdots \eta_{i-1}^{\frac{\alpha_{i-1}}{\alpha_i}} \eta_i \eta_{i+1}^{\frac{\alpha_{i+1}}{\alpha_i}} \cdots \eta_n^{\frac{\alpha_n}{\alpha_i}}\right)^{k_i} \cdots \eta_n^{k_n}$$

$$= \eta_1^{m_1 - \frac{\alpha_1}{\alpha_i}} \cdots \eta_{i-1}^{m_{i-1} - \frac{\alpha_{i-1}}{\alpha_i}} \eta_i^{m_i} \eta_{i+1}^{m_{i+1} - \frac{\alpha_{i+1}}{\alpha_i}} \cdots \eta_n^{m_n - \frac{\alpha_n}{\alpha_i}}.$$

By equating the corresponding exponents of η_i $(i = 1, \ldots, n)$, we get a system of linear equations with respect to the unknowns $k_1, k_2, \ldots, k_n$. Denote $x_1 = k_1 + 1$, $x_2 = k_2 + 1, \ldots, x_n = k_n + 1$. Then the equation takes the following form:

$$\begin{cases} x_1 + & +\alpha_1/\alpha_i x_i & = m_1 + 1 \\ x_2 + & +\alpha_2/\alpha_i x_i & = m_2 + 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & x_i & = m_i + 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_n/\alpha_i x_i + & +x_n = m_n + 1. \end{cases} \qquad (9)$$

If a polyhedron $\mathfrak{M}$ has one vertex of anisotropicity α, than as it is shown in [12], there is a facet with the outer normal μ^{i_0} such that the system of linear Eqs. (9) has a non-negative solution after the substitution $\xi = v^{-\mu^{i_0}}\eta$. Suppose after the substitution $\xi = v^{-\mu^{i_0}}\eta$, the system (9) has a non-negative solution. If in the obtained polyhedron $\mathfrak{M}$ the sought-for facet corresponding to the substitution i_0 is not the facet $(l^1, l^2, \ldots, l^{n-1}, \alpha)$, then, since this facet is as well a facet of the polyhedron $\mathfrak{N}$, the problem of finding the sought-for facet is resolved. If the sought-for facet is $(l^1, l^2, \ldots, l^{n-1}, \alpha)$, which is not a facet of the polyhedron $\mathfrak{N}$, then we consider the following n pieces $(n-1)$-dimensional facets of the polyhedron $\mathfrak{N}$, passing through the vertices $\{l^1, l^2, \ldots, l^{n-1}, \beta\}$, $\{l^1, l^2, \ldots, l^{n-2}, \beta, \alpha\}$, $\{l^1, l^2, \ldots, l^{n-3}, \beta, l^{n-1}, \alpha\}$, $\ldots, \{l^1, \beta, l^3, \ldots, l^{n-1}, \alpha\}$, $\{\beta, l^2, \ldots, l^{n-1}, \alpha\}$. Then we prove, that at least for one of them one of the following n systems of linear equations has a non-negative solution:

$$\begin{cases} x_1 + & +\beta_1/\beta_i x_i + & +\alpha_1/\alpha_n x_n = m_1 + 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & x_i + & +\alpha_i/\alpha_n x_n = m_i + 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \beta_n/\beta_i x_i + & + x_n = m_n + 1. \end{cases} \qquad (10)$$

Not that for $i = 1, \ldots, n$ the variables with coefficients of the form $\dfrac{\alpha_i}{\alpha_n}$ $(i = 1, \ldots, n)$ are zeros in (10).

Suppose $S = \{\alpha^1, \alpha^2, \ldots, \alpha^{n+2}\} \subset \mathbb{R}^n$ and denote

$$cone(S) = \left\{ \sum_{i=1}^{n+2} c_i \alpha^i \mid n \geq -1, \alpha^i \in S, c_i \in R, c_i \geq 0 \right\}.$$

For $S_0 = \{l^1, l^2, \ldots, l^{n-1}, \alpha\} \subseteq \mathbb{R}^n$ we denote

$$\Delta_0 = \Delta(l^1, l^2, \ldots, l^{n-1}, \alpha) = \begin{vmatrix} 1 & 0 & \cdots & 0 & \alpha_1/\alpha_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{n-1}/\alpha_n \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Let $\beta \in \mathbb{R}^n$.

For $S_i = (S_0 \setminus \{l^i\}) \bigcup \{\beta\}$ $(i = 1, \ldots, n-1)$, $S_n = (S_0 \setminus \{\alpha\}) \bigcup \{\beta\}$ we have

$$\Delta_i = \Delta(l^1, \ldots, l^{i-1}, \beta, l^{i+1}, \ldots, l^{n-1}, \alpha) = \begin{vmatrix} 1 & \alpha_i/\alpha_n \\ \beta_n/\beta_i & 1 \end{vmatrix} = 1 - \frac{\alpha_i}{\alpha_n} \cdot \frac{\beta_n}{\beta_i} \quad (i = 1, \ldots n-1),$$

$$\Delta_n = \Delta(l^1, \ldots, l^{n-1}, \beta) = \begin{vmatrix} 1 & 0 & \cdots & 0 & \beta_1/\beta_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \beta_{n-1}/\beta_n \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Denote $I = \{1, 2, \ldots, n-1\}$. Let $I = I_1 \bigcup I_2$, where I_1 is the set of indices i, for which the signs of Δ_i and Δ_0 coincide and $I_2 = I \setminus I_1$.

First we prove that Δ_i and Δ_0 have the same sign for some $i \in I$. The equation of the hyper-plane that cut $\mathfrak{N}$ is $\Delta_p(x) = \Delta(x - l_1, l_2 - l_1, \ldots, l_{n-1} - l_1, \alpha - l_1)$. Denote by β the point that was left out of $\mathfrak{N}$ after the cut. Then it is obvious that β and 0 lie in different halfspaces, i.e. $\Delta_p(0) \cdot \Delta_p(\beta) < 0$. Notice that $-\Delta_p(0) = \Delta_0$ and, thus, $\Delta_p(\beta)$ and Δ_0 have the same sign. Using the properties of the determinant, it is easy to show that $\sum\limits_{i=1}^{n} \Delta_i = \Delta_p(\beta) - \Delta_p(0) = \Delta_p(\beta) + \Delta_0$. Then since Δ_0 and $\Delta_p(\beta)$ have the same sign, their sum also has the same sign. Thus $\sum\limits_{i=1}^{n} \Delta_i$ has the same sign as Δ_0. So at least one of Δ_i has the same sign, as Δ_0.

So at least one of the systems of linear Eqs. (10) has a non-negative solution. Let $m + 1 \in cone(S_0)$, then there is a vector $m_0 = (m_{0,1}, \ldots, m_{0,n})$ with non-negative coefficients such that $m + 1 = \sum\limits_{j=1}^{n-1} m_{0,j} l^j + m_{0,n} \alpha$. Consider the system of linear Eqs. (10), for which $i \in I_1$. Let x_i be the solution to the i-th equation. Then by Cramer's Rule, using the properties of the determinant for $j < i$, we get

$$x_{i,j} = \Delta(l^1, l^2, \ldots, l^{j-1}, m+1, l^{j+1}, \ldots, l^{i-1}, \beta, l^{i+1}, \ldots, l^{n-1}, \alpha)$$
$$= m_{0,1} \Delta(l^1, l^2, \ldots, l^{j-1}, l^1, l^{j+1}, \ldots, l^{i-1}, \beta, l^{i+1}, \ldots, l^{n-1}, \alpha)$$
$$+ \cdots + m_{0,j} \Delta(l^1, l^2, \ldots, l^{j-1}, l^j, l^{j+1}, \ldots, l^{i-1}, \beta, l^{i+1}, \ldots, l^{n-1}, \alpha)$$
$$+ \cdots + m_{0,i} \Delta(l^1, l^2, \ldots, l^{j-1}, l^i, l^{j+1}, \ldots, l^{i-1}, \beta, l^{i+1}, \ldots, l^{n-1} \alpha)$$
$$+ \cdots + m_{0,n} \Delta(l^1, l^2, \ldots, l^{j-1}, \alpha, l^{j+1}, \ldots, l^{i-1}, \beta, l^{i+1}, \ldots, l^{n-1}, \alpha) = m_{0,j} + m_{0,i} \frac{\Delta_j}{\Delta_i}.$$

For $j > i$, the i-th and j-th rows are switched yielding the same result as for $j < i$. Using the properties of the determinant for $i = j$, we obtain

$$x_{i,i} = \Delta(l^1, l^2, ..., l^{i-1}, m+1, l^{i+1}, ..., l^{n-1}, \alpha) = m_{0,1}\Delta(l^1, l^2, ..., l^{i-1}, l^1, l^{i+1}, ..., l^{n-1}, \alpha) + \cdots +$$

$$+ m_{0,i}\Delta(l^1, l^2, ..., l^{i-1}, l^i, l^{i+1}, ..., l^{n-1}, \alpha) + \cdots + m_{0,n}\Delta(l^1, l^2, ..., l^{i-1}, \alpha, l^{i+1}, ..., l^{n-1}, \alpha)$$

$$= m_{0,i}\frac{\Delta_0}{\Delta_i}.$$

Note that $\frac{\Delta_0}{\Delta_i} > 0$ for $i \in I_1$ and $\frac{\Delta_j}{\Delta_i} \leq 0$ for $i \in I_1$ and $j \in I_2$, since the signs of Δ_j and Δ_i do not coincide and the signs of Δ_0 and Δ_i coincide and are non-zero. Thus we have $m_{0,j} \geq 0$, $x_{i,i} \geq 0$ and $x_{i,j} \geq 0$ for any $i \in I_1$ and $j \in I_2$.

Now, we express $x_{i,j}$, where $i \neq j \in I_1$, in terms of $x_{i,i}$ and $x_{j,j}$ as follows:

$$x_{i,j} = m_{0,j} - m_{0,i}\frac{\Delta_j}{\Delta_i} = \frac{\Delta_j}{\Delta_0}\left(m_{0,j}\frac{\Delta_0}{\Delta_j} - m_{0,i}\frac{\Delta_0}{\Delta_i}\right) = \frac{\Delta_j}{\Delta_0}(x_{j,j} - x_{i,i}).$$

Let k be such that $x_{k,k} = \min_{i \in I_1} x_{i,i}$. We claim, that the vector x_k is non-negative and we only need to show that $x_{k,j} \geq 0$, $k \neq j \in I_1$. Due to the choice of k we have $x_{j,j} - x_{k,k} \geq 0$ for $j \in I_1$ and $\frac{\Delta_j}{\Delta_0} > 0$ and so $x_{k,j} = \frac{\Delta_j}{\Delta_0}(x_{j,j} - x_{k,k}) \geq 0$.

For $i = n$ the system (10) also has a non-negative solution, since $x_{i,j} = m_{0,j} \geq 0$, $j = 1, \ldots, n$.

Thus, we showed that at least one of the system of linear Eqs. (10) has a non-negative solution. $\qquad\square$

C o r o l l a r y . It follows from the proof of Le'mma 1, that the power of logarithm in the inequality depends on the zeros of the system (10), i.e. instead of n in (5) we can take the maximum number of zeros of the system (10). In particular, if the system (10) has no zeros, then the constants $C_1, \ldots, C_n$ can be taken to be zeros.

Multianisotropic Kernels and Their Properties. Consider the multianisotropic kernels (3), denoting

$$\hat{G}_{1,j}(t, v) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i(t,\xi)} (2k)\left(v\xi^{\alpha^j}\right)^{2k-1} e^{-P(v,\xi)} d\xi.$$

We study the properties of $\hat{G}_0, \hat{G}_{1,j} \in S$.

Let μ^i ($i = 1, \ldots, n$) be the outer normal of the $(n-1)$-dimensional face, which is passing through the vertices $\{\alpha^1, \alpha^2, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^n, \sigma\}$, where σ is or α^{n+1}, or α^{n+2}. Denote by $\gamma = (\gamma^1, \ldots, \gamma^n)$ the point of intersection of the hyper-plane with the outer normals $\mu^1, \ldots, \mu^n$. Suppose the relation $\gamma_1 < \gamma_2 < \cdots < \gamma_{n-r} \leq \gamma_{n-r+1} \leq \cdots \leq \gamma_n$ ($r = 0, 1, \ldots, n-1$) holds between the coordinates of the vector γ.

As in [12], we build a set of vectors $\mathfrak{U}$ and $\mathfrak{B}$. Then, when $0 < v < 1$ for $\hat{G}_{1,j}(t, v)$ we have the following results (the proof is similar to the proof of Lemma 1.1 and 1.2 in [12]):

L e m m a 2. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_{n-r} \leq \gamma_{n-r+1} \leq \cdots \leq \gamma_n$. Then for any multi-index m and any even number N, for which $N\mathfrak{U}$ has only even coordinates, there exist constants C_i ($i = 0, 1, \ldots, n$) such that for any v, $0 < v < 1$,

$$|D^m\hat{G}_{1,j}| \leq \frac{v^{-\max\limits_{i=1,\ldots,2n-1}|\mu^i|}(C_n|\ln v|^n + \cdots + C_0)}{(1 + v^{-N}(t^{N\gamma} + t^{N\rho} + \cdots + t^{N\delta}))\ldots(1 + v^{-N}(t^{N\gamma} + \cdots + t^{N\sigma}))}, \qquad (11)$$

where out of all subsets of $\mathfrak{U}$ corresponding to $\mathfrak{B}$ the one present in the multipliers of the inequality (11) is the smallest.

L e m m a 3. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_{n-r} \le \gamma_{n-r+1} \le \cdots \le \gamma_n$. Then there exist numbers a_i $(i = 0, \ldots, l)$ and a natural number N_0 such that for any v, $0 < v < 1$, and $N > N_0$ the following inequality holds:

$$\int_0^\infty (n) \int_0^\infty \frac{dt_1 \ldots dt_n}{(1 + v^{-N}(t^{N\gamma} + t^{N\rho} + \cdots + t^{N\delta})) \cdots (1 + v^{-N}(t^{N\gamma} + \cdots + t^{N\sigma}))}$$

$$\le v^{\min_{i=1,\ldots,r} |\mu^i|} \left(a_l |\ln v|^l + \cdots + a_0 \right), \tag{12}$$

where l $(l \le r)$ is the number of equalities among the coordinates of the vector $\gamma = (\gamma_1, \ldots, \gamma_n)$.

Regularization of a Function and Integral Representation Through a Multi-anisotropic Kernels. For any measurable function f consider the regularization with the kernel $\hat{G}_0(t, v)$:

$$f_v(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(t) \hat{G}_0(t - x, v) dt, \quad x \in \mathbb{R}^n. \tag{13}$$

Then the function f_v will have the usual properties of the regularization, i.e.

L e m m a 4. If $f \in L_p(\mathbb{R}^n)$, $1 < p < \infty$, then $f_v \in L_p(\mathbb{R}^n)$, $\|f_v\|_{L_p(\mathbb{R}^n)} \to 0$ as $v \to \infty$ and $\lim_{v \to 0} \|f_v - f\|_{L_p(\mathbb{R}^n)} = 0$.

As in [12] (see Theorem 3.1), we can prove the following theorem of integral representation using the regularization (13):

T h e o r e m 1. Let a function f have the weak derivatives $D^{\alpha^i} f$, where $i = 1, \ldots, n+2$, α^i are the vertices of a completely regular polyhedron $\mathfrak{N}$ and $D^{\alpha^i} f \in L_p(\mathbb{R}^n)$, $i = 1, \ldots, n+2$, $1 \le p < \infty$. Then for almost all $x \in \mathbb{R}^n$ it has the representation

$$f(x) = f_h(x) + \lim_{\varepsilon \to 0} \sum_{i=1}^{n+1} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\varepsilon^h dv \int_{\mathbb{R}^n} D^{\alpha^i} f(t) \hat{G}_{1,i}(t - x, v) dt. \tag{14}$$

Embedding Theorems for Multianisotropic Spaces. Denote $W_p^{\mathfrak{N}}(\mathbb{R}^n) = \{f : f \in L_p(\mathbb{R}^n), \ D^{\alpha^i} f \in L_p(\mathbb{R}^n), \ i = 1, \ldots, n+2\}$ and call the multianisotropic Sobolev space with the norm

$$\|f\|_{W_p^{\mathfrak{N}}(R^n)} = \sum_{i=1}^{n+2} \left\| D^{\alpha^i} f \right\|_{L_p}.$$

Using the results of the previous section we can prove the following embedding theorem for multianisotropic spaces with two vertices of anisotropicity (for multianisotropic spaces with one vertex of anisotropicity see [12] , Theorem 4.2):

T h e o r e m 2. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_{n-r} \le \gamma_{n-r+1} \le \cdots \le \gamma_n$, p and q $(p \le q)$ be numbers such that $1 < p \le \infty$ or $1 \le p < \infty$ and $q = \infty$, $m = (m_1, m_2, \ldots, m_n)$ be a multi-index. Denote $\chi = \max_{i=1,\ldots,n+2} \left(|\mu^i| + (m, \mu^i) \right) - \min_{i=1,\ldots,r} |\mu^i| \left(1 - \frac{1}{p} + \frac{1}{q} \right)$.

If $\chi < 1$, then $D^m W_p^{\mathfrak{N}}(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n)$, i.e. any function $f \in W_p^{\mathfrak{N}}(\mathbb{R}^n)$ has a weak derivative $D^m f \in L_q(\mathbb{R}^n)$ and for any $h > 0$ the following inequality holds:

$$\|D^m f\|_{L_q(\mathbb{R}^n)} \le h^{1-\chi} \left(a_{l+n} |\ln h|^{l+n} + \cdots + a_0 \right) \sum_{i=1}^{n+2} \left\| D^{\alpha^i} f \right\|_{L_p(\mathbb{R}^n)}$$

$$+ h^{-\chi} \left(b_{l+n} |\ln h|^{l+n} + \cdots + b_0 \right) \|f\|_{L_p(\mathbb{R}^n)}, \tag{15}$$

for some constants $a_0, \ldots, a_{l+n}, b_0, \ldots, b_{l+n}$, and l $(l \le r)$ is the number of equalities between the coordinates of the vector $\gamma = (\gamma_1, \ldots, \gamma_n)$.

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