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M a t h e m a t i c s

ON SOLVABILITY OF AN INFINITE NONLINEAR SYSTEM OF ALGEBRAIC EQUATIONS WITH TEOPLITZ–HANKEL MATRICES

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In the present paper a special class of infinite nonlinear system of algebraic equations with Teoplitz–Hankel matrices is studied. Above mentioned class of equations has direct applications in radiative transfer theory. Existence componentwise positive solutions for the system in space l_1 are proved and some examples for mentioned equations, representing separate interest are given.

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Introduction and Statement of the Basic Results. The paper studies issues on solvability in space l_1 infinite nonlinear algebraic system of equations

$$x_n=\sum_{j=0}^{\infty}a_{n-j}h_j(x_j)+\sum_{j=1}^{\infty}a_{n+j}h_j^*(x_j),\ n=0,1,2\ldots, \tag{1}$$

with respect to the unknown infinite vector $x=(x_0,x_1,\ldots x_n,\ldots)^T$ (T is the sign of transposition).

The Teoplitz $A\equiv(a_{n-j})_{n,j=0}^{\infty}$ and the Hankel $B\equiv(a_{n+j})_{n,j=0}^{\infty}$ matrices satisfy the conditions

$$a_{-j}=a_j,\quad \forall\, j\in\mathbb{N}\cup\{0\},\ a_n>0,\ \forall\, n\in\mathbb{Z}, \tag{2}$$

$$\sum_{i=-\infty}^{\infty}a_i=1,\quad \sum_{j=0}^{\infty}j^2a_j<+\infty, \tag{3}$$

$$a_{n+1}<a_n,\quad \forall n\in\mathbb{N}\cup\{0\}. \tag{4}$$

The system (1) arises in discrete problems of radiative transfer theory (see [1, 2]). Such type of system also arises in kinetic theory of gases and p-adic string theory (see [3–5]).

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The papers [6–9] studied system of equations (1) with the conditions (3),(4) and $\nu(A) \equiv \sum\limits_{j=-\infty}^{\infty} ja_j < 0$ for the different restrictions on $\{h_j(u)\}_{j=0}^{\infty}$ and $\{h_j^*(u)\}_{j=0}^{\infty}$.

In the present paper we will construct componentwise positive l_1 solution for nonlinear system (1) in the symmetric case (see condition (2)) .

The basic result of present note is the following:

Theorem 1. **(Basic).** Let conditions (2)–(4) are satisfied and there exist numbers $\alpha \in (0, 1/2]$ and $\eta \in (0,1)$ such that:

a) for each fixed $j \in \mathbb{N} \cup \{0\}$ the functions $h_j(u)$ and $h_j^*(u)$ are increasing on the interval $[P_j(\eta), 1]$,

$$P_j(\eta) \equiv \eta \sum_{m=j+1}^{\infty} a_m, \quad j \in \mathbb{N} \cup \{0\}; \tag{5}$$

b) $h_j, h_j^* \in C[P_j(\eta), 1], \quad j = 0, 1, 2, \dots$;

c) the following inequalities are satisfied

$$0 \le h_j(u) \le 1 - (1-u)^{\alpha}, \ u \in [P_j(\eta), 1], \quad j = 0, 1, 2, \dots ; \tag{6}$$

$$h_j^*(P_j(\eta)) \ge \eta, \ h_j^*(1) \le 1, \quad j = 0, 1, 2, \dots \tag{7}$$

Then system (1) has componentwise positive solution in space l_1, i.e there exists $x = (x_0, x_1, x_2, \dots x_n, \dots)^T$, satisfying (1). Moreover $x_j > 0$, $\forall j \in \mathbb{N} \cup \{0\}$, and $\sum\limits_{j=0}^{\infty} x_j < +\infty$.

Auxiliary Facts. Together with equation (1) we consider the following nonlinear auxiliary system:

$$s_n = \sum_{j=0}^{\infty} (a_{n-j} - a_{n+j}) s_j^{\alpha}, \quad n \in \mathbb{N} \cup \{0\}, \tag{8}$$

with respect to the unknown infinite vector $S = (s_0, s_1, ..., s_n, ...)^T$, where the sequence $\{a_n\}_{n=-\infty}^{\infty}$ satisfies conditions (2)–(4). We consider the following iteration:

$$\begin{gathered} s_n^{(p+1)} = \sum_{j=0}^{\infty} (a_{n-j} - a_{n+j}) \left(s_j^{(p)}\right)^{\alpha}, \\ s_n^{(0)} \equiv 1, \quad n = 0, 1, 2 \dots, \quad p = 0, 1, 2, \dots \end{gathered} \tag{9}$$

Using (2)–(4), one can easily verify by induction that

$$s_n^{(p)} \ge 0, \quad n = 0, 1, 2, \dots, \quad p = 0, 1, 2, \dots, \tag{10}$$

$$s_n^{(p)} \downarrow \ \text{in } p, \quad n = 0, 1, 2, \dots, \tag{11}$$

i.e.

$$\forall n \in \mathbb{N} \cup \{0\}, \quad 0 \le s_n^{(p+1)} \le s_n^{(p)}, \quad p = 0, 1, 2, \dots \tag{12}$$

Below we prove by induction that

$$\overline{\mathbf{1}} - S^{(p)} \in l_1, \quad p = 0, 1, 2, \dots, \tag{13}$$

where $\overline{\mathbf{1}} = (1,1,\dots,1,\dots)^T$, $S^{(p)} = (s_0^{(p)}, s_1^{(p)}, \dots, s_n^{(p)}, \dots)^T$. For $p=0$, inclusion (13) immediately follows from $\overline{\mathbf{1}} - S^{(0)} = (0,0,\dots,0,\dots)^T$. Let (13) takes place for some natural p. Then, taking into account (2)–(4) and (10)–(11), from (9) we obtain

$$0 \le 1 - s_n^{(p+1)} = \sum_{j=0}^{\infty} a_{n-j} + \sum_{j=1}^{\infty} a_{n+j} - \sum_{j=0}^{\infty} a_{n-j} \left(s_j^{(p)}\right)^{\alpha} + \sum_{j=0}^{\infty} a_{n+j} \left(s_j^{(p)}\right)^{\alpha} \le$$

$$\sum_{j=0}^{\infty} a_{n-j} \left(1 - \left(s_j^{(p)}\right)^{\alpha}\right) + \sum_{m=n+1}^{\infty} a_m + \sum_{m=n}^{\infty} a_m \le \sum_{j=0}^{\infty} a_{n-j} \left(1 - s_j^{(p)}\right) + 2 \sum_{m=n+1}^{\infty} a_m + a_n.$$

According to induction assumption (2), (3), from the obtained estimation we get

$$\sum_{n=0}^{\infty} \left(1 - s_n^{(p+1)}\right) \le \sum_{j=0}^{\infty} \left(1 - s_j^{(p)}\right) + 2 \sum_{m=1}^{\infty} m a_m + \sum_{n=0}^{\infty} a_n < +\infty.$$

Therefore, $\overline{\mathbf{1}} - S^{(p+1)} \in l_1$. Thus the inclusion (13) is proved.

Now consider the following auxiliary infinite linear homogeneous algebraic system of equations:

$$\tau_n = \frac{2}{1+a_0} \sum_{j=n}^{\infty} (a_{j-n} - a_{j+n}) \tau_j, \quad n = 0,1,2,\dots, \tag{14}$$

with respect to the unknown infinite vector $\tau = (\tau_0, \tau_1, \dots, \tau_n, \dots)^T$, where sequence $\{a_i\}_{i=-\infty}^{\infty}$ satisfies conditions (2)–(4). We prove below that system (14) besides of trivial solution $(0,0,\dots,0,\dots)^T$, provides also a componentwise non-negative non-trivial solution in the space of bounded sequences. First we consider following non-homogeneous system:

$$q_n = \rho_n + \frac{2}{1+a_0} \sum_{j=n}^{\infty} (a_{j-n} - a_{j+n}) q_j, \quad n = 0,1,2,\dots, \tag{15}$$

with respect to the infinite vector $q = (q_0, q_1, \dots, q_n, \dots)^T$, where

$$\rho_n \equiv \frac{2}{1+a_0} \sum_{m=2n}^{\infty} a_m, \quad n = 0,1,2,\dots \tag{16}$$

Direct verification shows, that the vector $\overline{\mathbf{1}} = (1,1,\dots,1,\dots)^T$ satisfies (15). Now we verify that system (15), besides of the trivial solution $\overline{\mathbf{1}}$ also has a positive solution in space l_1.

We consider the following "majoranting" linear system

$$\alpha_n = \rho_n + \frac{2}{1+a_0} \sum_{j=n}^{\infty} a_{j-n} \alpha_j, \quad n = 0,1,2,\dots, \tag{17}$$

with respect to the vector $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n, \dots)^T$. From (2)–(4) it follows that

$$\frac{2}{1+a_0} \sum_{j=0}^{\infty} a_j = 1, \tag{18}$$

$$\begin{aligned} \nu(\rho) \equiv \sum_{n=0}^{\infty} n \rho_n \le \frac{2}{1+a_0} \sum_{n=0}^{\infty} n \sum_{m=n}^{\infty} a_m = \frac{2}{1+a_0} \sum_{m=0}^{\infty} a_m \sum_{n=0}^{m} n = \\ = \frac{1}{1+a_0} \sum_{m=0}^{\infty} m(m+1) a_m < +\infty, \quad \rho = (\rho_0, \rho_1, \dots, \rho_n, \dots)^T. \end{aligned} \tag{19}$$

Therefore, from results of work [10] it follows that system (17) has a positive solution in l_1. We consider the following iteration:

$$q_n^{(m+1)} = \rho_n + \frac{2}{1+a_0}\sum_{j=n}^{\infty}(a_{j-n}-a_{j+n})q_j^{(m)},$$
$$q_n^{(0)} = \rho_n, \quad m = 0,1,2,\ldots, \quad n = 0,1,2,\ldots \tag{20}$$

By induction in m it can be proved that for each $n \in \mathbb{N}\cup\{0\}$ the following relations hold:

$$q_n^{(m+1)} \geq q_n^{(m)}, \quad m = 0,1,2,\ldots, \tag{21}$$

$$q_n^{(m)} \leq \alpha_n, \quad m = 0,1,2,\ldots, \tag{22}$$

$$q_n^{(m)} \leq 1, \quad m = 0,1,2,\ldots \tag{23}$$

From (21)–(23), taking into account Weierstrass theorem, it follows that the sequence of vectors $q^{(m)} = (q_0^{(m)}, q_1^{(m)}, ..., q_n^{(m)}, ...)^T$ has limit: $\lim\limits_{m\to\infty} q^{(m)} = q \equiv (q_0, q_1, ..., q_n, ..)^T$, and moreover

$$\rho_n \leq q_n \leq \alpha_n, \ \ q_n \leq 1, \ \ n = 0,1,2\ldots \tag{24}$$

From (23), (2) and (3) it follows that the limit vector satisfies system (15). Thus system (15) besides of the trivial solution $\overline{\mathbf{1}} = (1,1,\ldots,1,\ldots)^T$ also has a componentwise positive solution $q = (q_0, q_1, \ldots, q_n, \ldots)^T \in l_1$, moreover the components of the vector q satisfy inequalities (24). Direct checking shows that the vector $\tau = \overline{\mathbf{1}} - q$ satisfies the system (14). From (24) it follows that

$$\tau_n \geq 0, \ \ \tau_n \not\equiv 0, \ \ \tau_n \leq 1, \ \ n = 0,1,2,\ldots, \tag{25}$$

since $q \in l_1$. Fixing this solution, we consider the following iteration:

$$\tau_n^{(m+1)} = \frac{2}{1+a_0}\sum_{j=n}^{\infty}(a_{j-n}-a_{j+n})\tau_j^{(m)}, \ \ \tau_n^{(0)} = 1, \ \ m = 0,1,2,..., \ \ n = 0,1,2,... \tag{26}$$

By induction by m we can check the accuracy of the following statements:

$$\tau_n^{(m+1)} \leq \tau_n^{(m)}, \ \ \tau_n^{(m)} \geq \tau_n, \quad n = 0,1,2,\ldots, \quad m = 0,1,2,\ldots \tag{27}$$

Rewriting iteration (26) in the form

$$\tau_n^{(m+1)} = \frac{2}{1+a_0}\sum_{i=0}^{\infty} a_i \tau_{n+i}^{(m)} - \frac{2}{1+a_0}\sum_{j=n}^{\infty} a_{j+n}\tau_j^{(m)}, \tau_n^{(0)} = 1, m = 0,1,2,..., n = 0,1,2,...$$

and using (2) and (4), it can be verified that for each $m \in \mathbb{N}\cup\{0\}$ we have:

$$\tau_{n+1}^{(m)} \geq \tau_n^{(m)}, \quad n = 0,1,2,\ldots \tag{28}$$

From (27) and (28) it follows that sequence of vector $\tau^{(m)} = (\tau_0^{(m)}, \tau_1^{(m)}, ..., \tau_n^{(m)}, ...)^T$, $m = 0,1,2,\ldots,$ has a limit as $m \to \infty$ that is $\lim\limits_{m\to\infty} \tau^{(m)} = \tau_n^*$, $n = 0,1,2,\ldots$ In addition the limit vector $\tau^* = (\tau_0^*, \tau_1^*, \ldots, \tau_n^*, \ldots)^T$ satisfies system (14) and satisfies the following properties:

$$\tau_n \leq \tau_n^* \leq 1, \quad n = 0,1,2,\ldots, \tag{29}$$

$$\tau_{n+1}^* \geq \tau_n^*, \quad n = 0,1,2,\ldots \tag{30}$$

Since $\tau_n \geq 0$, $\tau_n \not\equiv 0$, we have $\tau_n^* \geq 0$, $\tau_n^* \not\equiv 0$. Therefore, there exists number $n_0 \in \mathbb{N}$ such that $\tau_{n_0}^* > 0$. From (30) it follows for $n \geq n_0$ that

$$\tau_n^* \geq \tau_{n_0}^* > 0. \tag{31}$$

Return to the sequence of vectors $\left\{S^{(p)}\right\}_{p=0}^{\infty}$. Using the sequence approximation method, we prove that

$$s_n^{(p)} \geq \frac{\tau_n^*}{\sup\limits_{n\in\mathbb{N}\cup\{0\}} \tau_n^*} \left(\frac{1+a_0}{2}\right)^{\frac{1}{1-\alpha}}, \quad n = 0,1,2,\dots, \quad p = 0,1,2,\dots \tag{32}$$

In the case $p = 0$ the inequality (32) follows from the following chain of inequalities:

$$\frac{\tau_n^*}{\sup\limits_{n\in\mathbb{N}\cup\{0\}} \tau_n^*} \left(\frac{1+a_0}{2}\right)^{\frac{1}{1-\alpha}} \leq \left(\frac{1+a_0}{2}\right)^{\frac{1}{1-\alpha}} \leq 1 = s_n^{(0)}.$$

Let inequality (32) holds for some natural p. Then from (9), taking into account (2)–(4) and (14), we get

$$s_n^{(p+1)} \geq \sum_{j=0}^{\infty}(a_{n-j} - a_{n+j}) \left(\frac{\tau_j^*}{\sup\limits_{n\in\mathbb{N}\cup\{0\}} \tau_n^*}\right)^{\alpha} \left(\frac{1+a_0}{2}\right)^{\frac{\alpha}{1-\alpha}} \geq$$

$$\geq \left(\frac{1+a_0}{2}\right)^{\frac{\alpha}{1-\alpha}} \left(\sup_{n\in\mathbb{N}\cup\{0\}} \tau_n^*\right)^{-\alpha} \sum_{j=n}^{\infty}(a_{n-j} - a_{n+j})(\tau_j^*)^{\alpha} \geq$$

$$\geq \left(\frac{1+a_0}{2}\right)^{\frac{\alpha}{1-\alpha}} \left(\sup_{n\in\mathbb{N}\cup\{0\}} \tau_n^*\right)^{-1} \sum_{j=n}^{\infty}(a_{n-j} - a_{n+j})\tau_j^* =$$

$$= \left(\frac{1+a_0}{2}\right)^{\frac{\alpha}{1-\alpha}} \cdot \frac{1+a_0}{2} \cdot \frac{\tau_n^*}{\sup\limits_{n\in\mathbb{N}\cup\{0\}} \tau_n^*} = \left(\frac{1+a_0}{2}\right)^{\frac{1}{1-\alpha}} \frac{\tau_n^*}{\sup\limits_{n\in\mathbb{N}\cup\{0\}} \tau_n^*}.$$

From (11) and (32) it follows that sequence of vectors $S^{(p)} = (s_0^{(p)}, s_1^{(p)}, \dots, s_n^{(p)}, \dots)^T$, $p = 0,1,2,\dots$, has a limit as $p \to \infty$, i.e. $\lim\limits_{p\to\infty} S^{(p)} = S = (s_0, s_1, \dots, s_n, \dots)^T$, then the limit vector S satisfies (8) and the inequality

$$\left(\frac{1+a_0}{2}\right)^{\frac{1}{1-\alpha}} \frac{\tau_n^*}{\sup\limits_{n\in\mathbb{N}\cup\{0\}} \tau_n^*} \leq s_n \leq 1, \quad n \in \mathbb{N}\cup\{0\}. \tag{33}$$

From (33) and (31) for $n \geq n_0$ it also follows that

$$s_n \geq \left(\frac{1+a_0}{2}\right)^{\frac{1}{1-\alpha}} \frac{\tau_{n_0}^*}{\sup\limits_{n\in\mathbb{N}\cup\{0\}} \tau_n^*} > 0. \tag{34}$$

Thus proves the following:

Theorem 2. Let the sequence $\{a_i\}_{i=-\infty}^{\infty}$ satisfy condition (2)–(4), and $\alpha \in \left(0, \frac{1}{2}\right]$. Then the system (8) has a componentwise non-negative solution $S = (s_0, s_1, \ldots, s_n, \ldots)^T$ in space of bounded sequences. Moreover each component of the vector S satisfies inequalities (33) and (34).

In the following theorem we will additionally prove that

$$\overline{1} - S \in l_1. \tag{35}$$

Theorem 3. Let the conditions of the Theorem 2 are satisfied. Then the constructed solution S satisfies additional property (35).

Proof. First note that from (2)–(4) follows that

$$\gamma \equiv \sum_{j=-\infty}^{n_0} a_j < 1. \tag{36}$$

Using inclusion (13) below, by induction we prove that

$$\sum_{n=0}^{\infty}(1 - s_n^{(p)}) \le \left(2\sum_{m=1}^{\infty} m a_m + \sum_{n=0}^{\infty} a_n\right)(1-\varepsilon)^{-1}, \tag{37}$$

where

$$\varepsilon \equiv \max(\gamma, \beta), \quad \beta \equiv \left(1 + \sqrt{\left(\frac{1+a_0}{2}\right)^{\frac{1}{1-\alpha}} \frac{\tau^*_{n_0+1}}{\sup\limits_{n \in \mathbb{N}\cup\{0\}} \tau^*_n}}\right)^{-1}. \tag{38}$$

In the case $p = 0$ the inequality (37) obviously follows from (9). Let (37) holds for some $p \in \mathbb{N}$. We prove (37) in the case $p+1$. Taking into account (2)–(4), (34), (36) from (9), we obtain:

$$\sum_{n=0}^{\infty}\left(1 - s_n^{(p+1)}\right) \le \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} a_{n-j}\left(1 - \left(s_j^{(p)}\right)^{\alpha}\right) + 2\sum_{m=1}^{\infty} m a_m + \sum_{n=0}^{\infty} a_n =$$

$$= 2\sum_{m=1}^{\infty} m a_m + \sum_{n=0}^{\infty} a_n + \sum_{j=0}^{\infty}\left(1 - \left(s_j^{(p)}\right)^{\alpha}\right)\sum_{n=0}^{\infty} a_{n-j} \le$$

$$\le 2\sum_{m=1}^{\infty} m a_m + \sum_{n=0}^{\infty} a_n + \sum_{j=0}^{\infty}\left(1 - \sqrt{s_j^{(p)}}\right)\sum_{m=-\infty}^{j} a_m =$$

$$= 2\sum_{m=1}^{\infty} m a_m + \sum_{n=0}^{\infty} a_n + \sum_{j=0}^{n_0}\left(1 - \sqrt{s_j^{(p)}}\right)\sum_{m=-\infty}^{j} a_m +$$

$$+ \sum_{j=n_0+1}^{\infty}\left(1 - \sqrt{s_j^{(p)}}\right)\sum_{m=-\infty}^{j} a_m \le 2\sum_{m=1}^{\infty} m a_m + \sum_{n=0}^{\infty} a_n + \gamma\sum_{j=0}^{n_0}\left(1 - \sqrt{s_j^{(p)}}\right) +$$

$$+ \sum_{j=n_0+1}^{\infty}\frac{1 - s_j^{(p)}}{1 + \sqrt{s_j^{(p)}}} \le 2\sum_{m=1}^{\infty} m a_m + \sum_{n=0}^{\infty} a_n + \gamma\sum_{j=0}^{n_0}\left(1 - s_j^{(p)}\right) + \beta\sum_{j=n_0+1}^{\infty}\left(1 - s_j^{(p)}\right) \le$$

$$\leq 2\sum_{m=1}^{\infty} ma_m + \sum_{n=0}^{\infty} a_n + \varepsilon\left(\sum_{j=0}^{n_0}\left(1-s_j^{(p)}\right) + \sum_{j=n_0+1}^{\infty}\left(1-s_j^{(p)}\right)\right) \leq$$

$$\leq 2\sum_{m=1}^{\infty} ma_m + \sum_{n=0}^{\infty} a_n + \varepsilon\sum_{j=0}^{\infty}\left(1-s_j^{(p+1)}\right) \;\Rightarrow$$

$$\Rightarrow \sum_{j=0}^{\infty}\left(1-s_j^{(p+1)}\right) \leq \left(2\sum_{m=1}^{\infty} ma_m + \sum_{n=0}^{\infty} a_n\right)(1-\varepsilon)^{-1}.$$

Thus inequality (37) is proved. Tending p to infinity in (37) completes the Proof of Theorem. □

Remark. It we note that the Theorems 2 and 3 are the discrete analogues of the results obtained earlier in [11].

Proof of the Basic Results. Examples of Functions $\{h_j\}_{j=0}^{\infty}$ and $\{h_j^*\}_{j=1}^{\infty}$. Consider the following iteration for the basic system (1)

$$x_n^{(p+1)} = \sum_{j=0}^{\infty} a_{n-j} h_j\left(x_j^{(p)}\right) + \sum_{j=1}^{\infty} a_{n+j} h_j^*\left(x_j^{(p)}\right), \qquad x_n^{(0)} = 1 - s_n, \quad p = 0,1,2,\ldots, \quad n = 0,1,2,\ldots, \tag{39}$$

where $S = (s_0, s_1, \ldots, s_n, \ldots)^T$ are the solution of the system (8) and satisfies properties (33)–(35). By induction we will prove that for each $n \in \mathbb{N} \cup \{0\}$:

$$x_n^{(p)} \downarrow \text{in } p, \tag{40}$$

$$x_n^{(p)} \geq P_n(\eta), \quad p = 0,1,2,\ldots, \tag{41}$$

where $P_n(\eta)$ is given by formula (5). First we verify that $x_n^{(1)} \leq x_n^{(0)}$ and $x_n^{(0)} \geq P_n(\eta)$, $n = 0,1,2,\ldots$ Indeed, from (8), taking into account (2)–(3), we have

$$x_n^{(0)} = 1 - s_n = \sum_{j=0}^{\infty} a_{n-j}(1-s_j^{\alpha}) + \sum_{j=n+1}^{\infty} a_j + \sum_{j=0}^{\infty} a_{n+j} s_j^{\alpha} \geq$$

$$\geq \sum_{j=n+1}^{\infty} a_j \geq \eta \sum_{j=n+1}^{\infty} a_j = P_n(\eta), \;\; x_n^{(0)} \leq 1.$$

Using conditions (6), (7), taking into account (2)–(4), from (39) we get

$$x_n^{(1)} \leq \sum_{j=0}^{\infty} a_{n-j}\left(1 - \left(1 - x_j^{(0)}\right)^{\alpha}\right) + \sum_{j=1}^{\infty} a_{n+j} h_j^*\left(x_j^{(0)}\right) \leq$$

$$\leq \sum_{j=0}^{\infty} a_{n-j}\left(1 - s_j^{\alpha}\right) + \sum_{j=1}^{\infty} a_{n+j} = 1 - \sum_{j=0}^{\infty} a_{n-j} s_j^{\alpha} + \sum_{j=0}^{\infty} a_{n+j} s_j^{\alpha} - \sum_{j=0}^{\infty} a_{n+j} s_j^{\alpha} =$$

$$= 1 - \sum_{j=0}^{\infty}(a_{n-j} - a_{n+j}) s_j^{\alpha} - \sum_{j=0}^{\infty} a_{n+j} s_j^{\alpha} \leq 1 - s_n\,.$$

Assuming that $x_n^{(p)} \leq x_n^{(p-1)}$ and $x_n^{(p)} \geq P_n(\eta)$, $n = 0,1,2,\ldots$, for some $p \in \mathbb{N}$ and taking into account the monotonicity $h_j(u)$, $j = 0,1,2,\ldots$, $h_j^*(u)$, $j = 1,2,\ldots$, on u, from (39) we obtain

$$x_n^{(p+1)} \leq \sum_{j=0}^{\infty} a_{n-j} h_j\left(x_j^{(p-1)}\right) + \sum_{j=1}^{\infty} a_{n+j} h_j^*\left(x_j^{(p-1)}\right) = x_n^{(p)},$$

$$x_n^{(p+1)} \geq \sum_{j=0}^{\infty} a_{n-j} h_j(P_j(\eta)) + \sum_{j=1}^{\infty} a_{n+j} h_j^*(P_j(\eta)) \geq$$

$$\geq \sum_{j=1}^{\infty} a_{n+j} h_j^*(P_j(\eta)) \geq \eta \sum_{j=1}^{\infty} a_{n+j} = P_n(\eta).$$

From (40) and (41) we obtain that the sequence of vectors $x^{(p)} = (x_0^{(p)}, x_1^{(p)}, \ldots, x_n^{(p)}, \ldots)^T$, $p = 0, 1, 2, \ldots$, has a limit as $p \to \infty$:

$$\lim_{p\to\infty} x^{(p)} = x = (x_0, x_1, \ldots, x_n, \ldots)^T.$$

From (40) and (41) it also follows that

$$P_n(\eta) \leq x_n \leq 1 - s_n, \quad n = 0, 1, 2, \ldots \tag{42}$$

Since $\overline{\mathbf{1}} - S \in l_1$, then from (42) we obtain that $x \in l_1$. Now we verify that vector x satisfies system (1). First we note that the sum of series

$$\sum_{j=0}^{\infty} a_{n-j} h_j(x_j) + \sum_{j=0}^{\infty} a_{n+j} h_j^*(x_j)$$

is bounded uniformly in n. Using (42), (6), (7) and (2)–(4), we get

$$\sum_{j=0}^{\infty} a_{n-j} h_j(x_j) + \sum_{j=1}^{\infty} a_{n+j} h_j^*(x_j) \leq \sum_{j=0}^{\infty} a_{n-j}(1 - (1 - x_j)^{\alpha}) +$$

$$+ \sum_{j=1}^{\infty} a_{n+j} h_j^*(1) \leq \sum_{j=0}^{\infty} a_{n-j} + \sum_{j=1}^{\infty} a_{n+j} = 1 < +\infty.$$

On the other hand, $h_j, h_j^* \in C[P_j(\eta), 1]$, $j = 0, 1, \ldots$ Hence, we can pass to the limit in the summation sign as $p \to \infty$. Indeed in (39) passing to the limit, taking into account above mentioned Remark, we obtain

$$x_n = \lim_{p\to\infty} x_n^{(p+1)} = \lim_{p\to\infty} \left(\sum_{j=0}^{\infty} a_{n-j} h_j \left(x_j^{(p)} \right) + \sum_{j=1}^{\infty} a_{n+j} h_j^* \left(x_j^{(p)} \right) \right) =$$

$$= \sum_{j=0}^{\infty} a_{n-j} \lim_{p\to\infty} \left(h_j \left(x_j^{(p)} \right) \right) + \sum_{j=1}^{\infty} a_{n+j} \lim_{p\to\infty} \left(h_j^* \left(x_j^{(p)} \right) \right) =$$

$$= \sum_{j=0}^{\infty} a_{n-j} h_j \left(\lim_{p\to\infty} x_j^{(p)} \right) + \sum_{j=1}^{\infty} a_{n+j} h_j^* \left(\lim_{p\to\infty} x_j^{(p)} \right) = \sum_{j=0}^{\infty} a_{n-j} h_j(x_j) + \sum_{j=1}^{\infty} a_{n+j} h_j^*(x_j).$$

Thus the Theorem 1 is proved.

At the end of the work we list some examples of functions $\{h_j\}_{j=0}^{\infty}$, $\{h_j^*\}_{j=1}^{\infty}$ satisfying all conditions of Basic Theorem 1:

1. $h_j(u) = \left(1 - (1-u)^{\alpha}\right) c_j$, where $\alpha \in \left(0, \frac{1}{2}\right]$, $0 < c_j \leq 1$, $j = 0, 1, 2, \ldots$, $u \in [P_j(\eta), 1]$,
   $h_j^*(u) = \dfrac{u}{u + P_j(1-\eta)}$, $\eta \in (0, 1)$, $u \in [P_j(\eta), 1]$, $j = 1, 2, 3, \ldots$;

2. $h_j(u) = \ln\left(2-(1-u)^{\alpha}\right) d_j(u)$, $\alpha \in \left(0, \frac{1}{2}\right]$, $0 < d_j(u) \le 1$, $u \in [P_j(\eta), 1]$, $d_j(u) \uparrow$ in u on $[P_j(\eta), 1]$, $d_j \in C[P_j(\eta), 1]$, $j = 0, 1, 2, \dots$,
$h_j^*(u) = \dfrac{u}{u + P_j(1-\eta)} + \dfrac{P_j(1-\eta)}{1+P_j(1-\eta)} u^r$, $r > 1$, $u \in [P_j(\eta), 1]$, $j = 1, 2, 3, \ldots$;

3. $h_j(u) = \dfrac{1-(1-u)^{\alpha} + \ln\left(2-(1-u)^{\alpha}\right)}{2} d_j(u)$, $u \in [P_j(\eta), 1]$, $j = 0, 1, 2, \ldots$,
$h_j^*(u) = \left(\left(\dfrac{u}{1+P_j(1-\eta)}\right)^2 + \dfrac{u^{r+1}P_j(1-\eta)}{(u+P_j(1-\eta))(1+P_j(1-\eta))}\right)^{\frac{1}{2}}$, $r > 1$,
$u \in [P_j(\eta), 1]$, $j = 1, 2, 3, \dots$

The examples of sequences $\left\{d_j(u)\right\}_{j=0}^{\infty}$ can serve the following functions:

1. $d_j(u) = 1 - \delta_j e^{-u}$;
2. $d_j(u) = \dfrac{\delta_j u}{1+u}$, $0 < \delta_j \le 1$, $j = 0, 1, 2, \dots$

Discuss same the details of Example 2.

Since $d_j(u) \uparrow$ in u *on* $[P_j(\eta), 1]$, $j = 0, 1, 2, \dots$, and

$$\frac{d}{du}\left(\ln\left(2-(1-u)^{\alpha}\right)\right) = \frac{1}{2-(1-u)^{\alpha}} \frac{\alpha}{(1-u)^{1-\alpha}} > 0\,,\ j = 0, 1, 2, \dots,$$

$$u \in [P_j(\eta), 1], h_j(u) \ge d_j(u) \ln 1 = 0\,,$$

$$\frac{dh_j^*(u)}{du} = \frac{P_j(1-\eta)}{(u+P_j(1-\eta))^2} + \frac{rP_j(1-\eta)}{1+P_j(1-\eta)} u^{r-1} > 0,\ \ j = 1, 2, \dots,\ u \in [P_j(\eta), 1]\,,$$

condition a) of Theorem 1 is satisfied. The condition b) is also satisfied, since the given functions are continuous on $[0, 1] \supset [P_j(\eta), 1]$, $\forall\ j = 0, 1, 2, \dots$ We cheek the condition c). Taking into account $\ln(1+x) \le x$, $x \ge 0$, we have

$$h_j(u) = \ln\left(1+1-(1-u)^{\alpha}\right) d_j(u) \le \ln\left(1+1-(1-u)^{\alpha}\right) \le 1-(1-u)^{\alpha}\,,$$

$$h_j(u) \ge d_j(u) \ln 1 = 0,$$

$$h_j^*(P_j(\eta)) = \frac{P_j(\eta)}{P_j(\eta)+P_j(1-\eta)} + \frac{P_j(1-\eta)}{1+P_j(1-\eta)} P_j^r(\eta) \ge$$

$$\ge \frac{\eta \sum\limits_{k=j+1}^{\infty} a_k}{\eta \sum\limits_{k=j+1}^{\infty} a_k + (1-\eta) \sum\limits_{k=j+1}^{\infty} a_k} = \eta,$$

$$h_j^*(1) = \frac{1}{1+P_j(1-\eta)} + \frac{P_j(1-\eta)}{1+P_j(1-\eta)} = 1.$$

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